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ABSTRACT

This document contains 15 case studies developed by the group teachers during a teaching experiment. Background information on each second grader, behavior and achievement characteristics, and analyses of the evaluation interviews are presented. (MS)

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PMDC Technical Report
No. 12

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Teaching Experiment: The Effect of Manipulatives in Second Graders' Learning of Mathematics Volume II, Case Studies

Edited by Merlyn J. Behr

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FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, question the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded; it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. Such is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out

[1] "Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-268.

[2] PMDC is supported by the National Science Foundation, Grant No. PES 74-18106-A03.

to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby, and Lauren Woodly. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee, consisting of Merlyn Behr (Chairman), Thomas Cooney, and Tom Denmark.

Eugene D. Nichols
Director of PMDC

PREFACE

The statement "Use manipulatives to teach mathematics" has been repeated so often and in so many different contexts that there is danger that the statement will begin to be accepted to the point where the use of manipulatives in teaching mathematics appears to be a panacea. There are, of course, no panaceas in the teaching of mathematics; yet, it is apparent that there are more questions that need to be answered about the why and how of using manipulatives to facilitate the learning of mathematics than most writers recognize.

While it may seem intuitively obvious that using manipulative aids to build a concrete conceptual base will in the final analysis help children to associate concepts and the symbolization of concepts in a meaningful way, it seems apparent that the "gap" between children's ability to perceive mathematics through manipulatives and their ability to associate symbolism with the concept is great. The question of how this gap is narrowed and finally closed in a problem that has had very little investigation.

When children first start school, they come with certain intuitive notions about mathematics. Whether or not these intuitive notions are tapped and built upon by instruction in our schools is an open question. This is true because we know very little about what these intuitive notions of mathematics are. Children have various finger manipulation strategies for doing addition, for example. Teaching practices frequently discourage children from using these strategies, and indeed, seldom are these strategies extended and a relationship between children's intuitive strategies and school strategies developed.

This publication is intended to share with the reader the information obtained from a teaching experiment which dealt with the question, "What are some important variables which affect how well children learn from manipulative aids:" Information about the relative effectiveness of Dienes blocks, counting sticks, and an abacus for the teaching of place value concepts to second grade children was investigated. Also of interest in the experiment was the question of whether systematic use of all three of the manipulatives would prove to be more effective for learning these concepts than just one manipulative.

The approach used in this study was that of a teaching experiment. The concept of a teaching experiment employed was that of gaining practical and anecdotal data in a teaching-learning situation. The teaching involved a teacher working with a small group of children, rather than a normal-sized class. Because of the small groups, statistics presented in this study's statistical data must be interpreted with caution. Of more interest are the questions and hypotheses which are suggested by the investigation.

The research reported in Volumes I and II was an attempt to gain some insights about the question of what are significant variables related to

how to use manipulatives in teaching mathematics to children. The reader will soon observe that the research method employed was different from that of traditional education research. Very small groups of children were involved in what was considered a teaching experiment. This concept of a teaching experiment represents a first (or second) step in a research effort. The research reported herein is not hypothesis testing of the familiar research tradition. Instead, it is hypothesis generating--more of an attempt is made at clarifying problems for further investigation than at answering pre-stated questions.

The reader of the volumes will find data presented in various forms; a great deal more "raw data" is presented than is ordinarily done in a research report. This raw data is presented in such diverse forms as raw scores on tests, both written and clinical interview tests, summaries of child responses extracted from daily logs kept by teachers of the small groups, and, finally, a number of case studies of children involved in the experiment.

Volume I contains sections which describe the rationale and conduct of the experiment in detail. In addition, Volume I includes information about the results of the investigation. Volume II consists entirely of the fifteen case studies conducted by the group teachers. As background for Volume II, the reader should refer to Chapters I and III of Volume I.

Many individuals contributed to the conceptualization of this study. The contribution made by the PMDC Advisory Board, Staff, and Evaluator, who reacted to the initial proposal is gratefully acknowledged. Special thanks are due Cynthia Clarke, Patricia Campbell, Stewart Wood, Judy Voran, and Ella Barco, who served as group teachers in the teaching experiment, and Max Gerling, who supervised the videotaping of lessons and interviews. Thanks are also due the project administrative assistant, Janelle Hardy, publications editor, Maria Pitner, and typists, Mary Harrington, Julie Rhodes, and Joe Schmerler.

MERLYN BEHR
Principal Investigator

Professor Behr is on leave of absence from Northern Illinois University.

I. THE CASE OF B

by

Ella Barco

BIOGRAPHICAL AND TESTING INFORMATION

B is a male child, born September 15, 1968. He lives with his parents and a six year old brother. B's father is a student, and his mother is a practical nurse. B's family moved to this city recently, and B entered the second grade in the school at which the experiment was conducted.

B is a very outdoor oriented child who loves horses, camping, and motor cycles. He seems to be a relatively happy boy; however, he does not like school. B could never follow a task through to completion and did not possess independent work skills. His written work was very sloppy. He did not like discipline.

B was a good speller and bragged continuously about placing fourth in a recent spelling bee. B worried about competition and spent a great deal of time trying to see how much progress others were making on their tasks to the neglect of his own tasks.

B's entering second grade Otis-Lennon IQ was 97. On the Comprehensive Test of Basic Skills administered at grade 2.6, B had grade equivalent scores of 2.6, 2.8, and 2.7 on reading, language, and mathematics, respectively. B was also a "know it all" child who was not well liked by fellow classmates. He had a superiority complex and manifested racial prejudices on a number of occasions.

B had an entering KeyMath grade equivalent of 2.2. His counting skills were above average as he could identify missing numbers in a written sequence (__, 6, 7, __, 9; __ 19; 98, 99, __, 101) and could compare the numerousness of differing sets. B could not recognize counting by three's. He could add one- and two-digit numerals without regrouping and experienced success on one addition problem with regrouping (66 + 4). B could subtract problems involving a one-digit minuend and a one-digit subtrahend but he could not subtract problems involving a two-digit minuend or subtrahend without or with regrouping. He could perform mental computations for some problems (1 + 1; 2 + 2; 1 + 4 - 2) and solved five word problems involving the operations of addition and subtraction.

In May of the second grade, B was again administered the KeyMath test and received a grade equivalent score of 2.7. At this time, B was able to recognize counting by three's, and could also add and subtract problems involving one or two-digit numerals without or with regrouping. There was no noticeable improvement in his mental computational skills or ability to solve word problems.

On the PMDC second grade test in the fall, B could count from 35 to 46 and count back from 6 to 1. He could not count by tens from 10 to 130 or from 26 to 126. He could determine the number of a set represented by 6 bundles of ten straws but could not determine the number of a set represented by 3 bundles of

ten straws and 7 single straws. He could not count by tens and ones to determine the value of a display of objects or to form a set representing a particular number. B was able to use counters to solve an addition problem whose sum was six but could not use counters to solve an addition problem involving 1 two-digit addend and 1 one-digit addend. Neither was he able to use counters to solve subtraction problems. B made no attempt to solve missing addend problems. He answered 2 of 3 items correctly dealing with the concepts of more and less ($7 > 4$; $8 < 12$; $19 > 31$) and experienced no difficulty in ordering four numerals (2, 3, 5, and 9) from smallest to largest. B did not attempt to answer any questions which called for identifying names for the same numbers (6 + 3 and 5 + 4; 4 + 1 and 3 + 2; 6 - 1 and 3 + 2; 5 - 2 and 4 - 1).

During the spring of the second grade, B was given a PMDC retest. This time he was able to count from 10 to 130 but when asked to count from 26 to 126 only counted to 106 after skipping 66. He could also determine the number of a set represented by 3 bundles of ten straws and 7 single straws. B could also count by tens and ones to determine the value of a display of objects or to form a set representing a particular number. He could use counters to solve addition and subtraction problems involving one and two-digit numerals without or with regrouping. B solved all missing addend problems correctly and was successful on all items dealing with concepts of "greater than and less than." He also attempted and worked correctly all items dealing with identifying names for the same number.

As a second grader in the T1 teaching experiment, B was assigned to the embodiment group (U1) using sticks. Initially, B was easily distracted from his tasks and had to be reminded frequently to stay on tasks until completion. Eventually, B learned to work independently and became a good student. His work was often untidy but correct. Once a concept was learned, B did not care to continue work at the manipulative level and would say "I'm tired of working with sticks; when are we going to write?"

B frequently wanted to do worksheets without receiving instructions and always felt that he knew what to do but this proved not to be the case in many instances. He wanted to appear superior to others in the group and would often remark, "Aren't you finished yet?" or "That's easy" or "Didn't I do good work today, teacher?"

The following discussion of B's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction with regrouping, addition and subtraction without regrouping.

PLACE VALUE

Initially, B had difficulty responding to questions related to counting stick displays because he didn't pay attention and never knew what to do. It soon became apparent that when B listened to explanations and instructions, he could respond successfully to most questions. B had little difficulty counting picture and stick displays; however, he would make occasional careless errors. He was the first one in the group to read terms such as, "one, ones, ten, tens, bundle, bundles, stick and sticks."

B was not only untidy in his written work but made a number of careless errors in recording. He encountered difficulties in filling in blanks on worksheets containing exercises such as:

3 bundles and 5 sticks

___ tens and ___ ones

___ and ___.

B could usually fill in the second line correctly (__ tens and __ ones) but had difficulty determining what should go on the third line (__ and __). Eventually, he overcame this difficulty and had no trouble filling in similar worksheets at a later date.

B could tell how many in all for a display of 1-9 bundles and could form stick displays for numbers from 10-90 when such a number was given orally by the teacher. Given a two-digit numeral, B could explain the meaning of the digits verbally in terms of tens and tens and ones and could use sticks to explain the numerals. On wordsheets, he could write and tell the number of tens and ones that were in a two-digit numeral but had difficulty determining the total number of ones. B was one of the few children in the group to respond successfully to worksheet items such as:

45 = 4 tens + 5 ones

40 + 5

45.

In the spring of the year, B made a smooth transition to three-digit numerals between 100 and 200. B could count from 1-100 by ones independently. B had difficulties, however, determining how many tens were in a number such as 170 or in a picture display of a three-digit number. His response would often denote the number of ones rather than the number of tens. Samples of B's work illustrate these difficulties:

Picture Display:	B's Response:
1 Bundle of bundles and 5 bundles	150 tens
1 Bundle of bundles and 1 bundle	110 tens

These difficulties continued to plague B at the symbolic level as indicated by the following samples of his work:

6 tens, 1 hundred = 106 tens

120 tens = 2 tens

170 = 71 tens

B could count stick and picture displays for numbers greater than 200 and could form stick displays to represent given numbers. Generally, he had no difficulties in determining the number of hundreds, tens, and ones contained in a three-digit numeral.

B had no difficulties reading three-digit numerals and seldom made mistakes. He also had the ability to change a given display so that it would show the same number but with "more tens" or "more hundreds."

At the end of the school year during the final evaluation interview, B was able to transfer his concept of place value to new situations in that he was able to represent 2 three-digit addends correctly with beans, a manipulative that was not used during the T1 teaching experiment. B was also able to correctly represent three-digit numerals using sticks, a familiar manipulative. He could read three-digit numerals and determine the total number of tens contained in a given number. B was able to determine the value of 2 hundreds, 13 tens, 4 ones (334).

When shown 156 and a picture display depicting 156 arranged as 14 groups of tens and 16 ones, B was unable to relate the picture to the numeral and did not think that the picture went with the numeral. Neither did he think that "245" and "1 hundred, 13 tens, 15 ones" were names for the same number.

ORDERING

Ordering pictures representing numbers from 10 to 20 was difficult, initially, for B. When given a set of pictures to order depicting numbers from 11 to 20, B was the first to finish but he had ordered them incorrectly ("11, 12, 13, 14, 15, 19, 18, 17, 20"). Later, he was able to order them correctly. On another occasion, when given a set of pictures to order depicting numbers from 20 to 30, B ordered them in the following manner: "21, 22, 23, 24, 25, 26, 27, 28, 29, 30." He held the picture for 20 in his hand and seemed puzzled about its place in the sequence but eventually realized that it should precede 21. No problems were encountered by B in ordering picture sets for 40-50, 80-90 and 10-90.

B had difficulty reading phrases (" is less than" and "is greater than") and symbols ($<$, $>$). He had similar difficulties in constructing number sentences of the form:

____ "is greater than" ____

____ "is less than" ____

____ $>$ ____

____ $<$ ____

B, frequently, knew which of 2 numbers was the greater from picture or stick displays but had difficulties verbalizing or writing a formal statement to this effect. Eventually, B overcame these difficulties and learned to handle the formal statements in both written and verbal form. After appearing to understand concepts of "more" and " less" in comparing two numbers, B had trou-

ble determining whether a particular number sentence was true or false. For example, he knew that 59 was greater than 39 but could not tell whether the sentence "59 is greater than 39," was true or false.

After representing given numbers with stick displays, B could easily show sticks to represent 1 more, 10 more, 100 more, 1 less, 10 less, and 100 less. This knowledge, however, did not transfer to picture sets. B had a considerable amount of difficulty trying to find pictures from a picture set showing 10 more or 10 less than the value depicted by a particular picture stick display. For example, B selected the picture representing 343 when asked to find a picture representing 10 more than a picture representing 243. Selecting pictures to show 1 more, 1 less, 100 more, or 100 less than the value of a given picture was easier for B.

At the symbolic level, similar difficulties were observed when B had to show 10 more or 10 less than a given number. Showing 1 more, 1 less, 100 more or 100 less seemed to be an easier task.

During B's final evaluation interview, B's grasp of ordering was revealed when he correctly determined the number that was 10 more than 137 and the number that was 10 less than 253.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

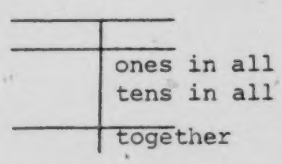
B had no difficulty showing the addition and subtraction of number pairs using stick displays and was able to verbalize what he had done. He could also write addends and sums for number sentences which were read orally by the teacher.

When working with pictures, B had no trouble selecting pictures for a given addition problem and was able to put the pictures in sequential order.

Initially, B had difficulty recording and making careless errors on worksheets. His work was usually very sloppy. He had a tendency to place tens under ones and vice versa but usually obtained the correct answers. Samples of his recording errors follow:

$$\begin{array}{r} 56 \\ -\ 42 \\ \hline 14 \end{array} \qquad \begin{array}{r} 37 \\ -\ 2 \\ \hline 35 \end{array}$$

With the passage of time, B improved his recording skills. He also experienced difficulties in filling blanks for worksheet items similar to the one below:



He knew where to place the addends and sum but often made errors in the intermediate step dealing with the "ones in all" and the "tens in all" step.

B did not care to use the form board to assist him in using manipulatives for working addition or subtraction problems. He felt that he knew what to do so he did not want to spend time manipulating.

During the teaching of addition and subtraction, B was frequently absent, but after a day of instruction, would have no difficulties performing tasks that others had worked on for several days. He knew quite a few basic facts from memory and perhaps used counting strategies for adding and subtracting less frequently than most of the other group members.

B did well on his addition and subtraction achievement tests. On parts of the test that were timed, B's time was longer than it should have been because he spent a great deal of his time looking around to see the progress of other group members. B worried a lot aobut the progress of other group members on most tasks. He liked to feel that he was top student on most tasks and would frequently reply when doing his work, "That's easy."

ADDITION AND SUBTRACTION WITH REGROUPING

B had no major problems adjusting to solving addition and subtraction problems with regrouping. When he first encountered the form board for addition with regrouping, B had difficulty displaying the correct number of sticks on the "tens in all" line. He wanted to count the ten he got as a result of trading as a part of the tens in all and would have 1 ten more than what he should have had. After a few enactive lessons on addition with regrouping, B overcame this problem. Iconic lessons and symbolic lessons did not pose any problems for B as far as addition with regrouping was concerned but B would make careless errors in basic facts as indicated by the following samples of his work:

$$\begin{array}{r} 19 \\ +24 \\ \hline 13 \\ 30 \\ \hline 33 \end{array} \qquad \begin{array}{r} 57 \\ +35 \\ \hline 16 \\ 80 \\ \hline 96 \end{array}$$

Subtraction problems with regrouping would present problems for B from time to time. Some errors were made due to B's failure to trade in problems where there were not enough ones to take away the required number of ones. For example, B's solution to 72 - 49 was written as follows:

$$\begin{array}{r} 72 \\ -49 \\ \hline 27 \end{array}$$

Other errors were made due to errors is basic facts as indicated by excerpts of his written work below:

$$\begin{array}{r} {}^{3}\not{4}\not{2}^{16} \\ -\ 7 \\ \hline 33 \end{array} \qquad \begin{array}{r} {}^{8}\not{9}\not{2}^{17} \\ -61 \\ \hline 26 \end{array} \qquad \begin{array}{r} {}^{}\not{8}\not{2}^{17} \\ -45 \\ \hline 49 \end{array}$$

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Still other errors were due to the fact that B decreased his tens but did not increase his ones. For example, he solved 48 - 39 in the following manner:

$$\begin{array}{r} 48 \\ -39 \\ \hline 11 \end{array}$$

Zeros in the minuend bothered B from time to time, and he would sometimes subtract a non-zero number from zero and get an answer of zero. An example of this error follows:

$$\begin{array}{r} 90 \\ -37 \\ \hline 67 \end{array}$$

On worksheets involving both addition and subtraction problems, B was careless in reading his operation signs and would sometimes add when he should have subtracted and vice versa.

During the final evaluation interview, B was able to solve the following three-digit addition problem with a familiar manipulative (sticks):

$$\begin{array}{r} 345 \\ +173 \\ \hline 518 \end{array}$$

This represented a new situation for B because he had not been taught regrouping involving three-digit numerals. He also solved the following three-digit problem using paper-pencil:

$$\begin{array}{r} 426 \\ +182 \\ \hline 608 \end{array}$$

B was also able to transfer his concept of regrouping to a new manipulative (beans) which he had not been instructed to use and solved the following problems correctly:

$$\begin{array}{r} 324 \\ +195 \\ \hline 519 \end{array} \qquad \begin{array}{r} 436 \\ -175 \\ \hline 261 \end{array}$$

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B's concept of regrouping did not enable him to solve the following subtraction problems correctly.

With sticks:

$$\begin{array}{r} 245 \\ -148 \\ \hline 98 \end{array}$$

(B's answer)

Paper-Pencil:

$$\begin{array}{r} {}^{1}\cancel{2}\,{}^{13}\cancel{3}\,{}^{17}\cancel{5} \\ -176 \\ \hline 69 \end{array}$$

II. THE CASE OF C

by

Ella Barco

BIOGRAPHICAL AND TESTING INFORMATION

C is a male child born March 22, 1968. He lives in a house with his grandparents and a semi-retarded uncle who assists the grandmother occasionally in caring for him. C has a nine year old sister who lives with him and a mother who lives out of town. The grandmother tries to keep the family together. C's mother visits him from time to time, but he is happiest when his mother is not there.

In the first grade, C would relate to only one teacher and was a behavior problem whenever other teachers were around. He was a loner who would play "at" people but not "with" them. He wanted to dominate other children and would often strike them for no apparent reason. C had a cruel streak and would often talk about kicking animals and pulling tails off of dogs. Sometimes C would punch the teacher with his fists and would continue to do so until he was reprimanded. Often times, he called it playing but never knew where to draw the line. C seemed to scream out for attention.

C was very emotional but refused to show his emotion. When he was disciplined, he would often show tears but never cry and would clam up and refuse to talk about his problems.

C handled money on several occasions that his grandmother was not aware that he had. It seemed as if he had used money earmarked for school supplies to pay off children on the way to school to keep from being beaten up.

C held most things inside and seldom talked about his inner feelings and rarely smiled. He had a poor self-concept based on a Self-Concept Test administered by the guidance counselor. His test responses indicated that he liked his teacher but didn't think the teacher liked him. It also revealed that he was half happy with himself and half sad with himself.

C did not want to be thought of as a "good" person but preferred to be labeled a "bad" person. He detested stars for good deportment or achievement. C seemed to enjoy TV and would watch it very attentively. Academically, C's performance was below average in the first grade. With the proper motivation and interest C could have worked on an average or above level in most areas according to his teacher. On the Comprehensive Test of Basic Skills (CTBS) administered at grade 1.6, C's grade equivalent scores were 1.4, 1.5 and .8 for reading, language, and mathematics, respectively.

During second grade, C made considerable progress in his academic studies. His entering second grade Otis-Lennon IQ was 81. On the CTBS administered at grade 2.6, C had a grade equivalent scores of 2.8, 2.6 and 2.6 in reading, language, and mathematics, respectively.

C had an entering KeyMath grade equivalent of 1.8. He could compare the numerousness of sets of nine or fewer objects and could find the missing number in a sequence such as 1, 2, __, 4, 5. He could not identify missing numbers for sequences such as: __, 6, 7, __, 9; __, 19,; 98, 99, __, 101. He could add 2 one-digit addends involving sums less than ten but could not add problems involving sums greater than ten without or with regrouping. Neither could he solve missing addend problems. He could not do subtraction problems involving minuends greater than five and subtrahends greater than two. The only mental computation problems that C could handle were those involving a sum less than five. He could solve five word problems involving the operations of addition and subtraction.

In May of the second grade, C was again administered the KeyMath test and received a grade equivalent score of 2.5. At this time, C was able to identify missing numbers in a written sequence such as __, 6, 7, __, 9; __, 19; 98, 99, __, 101. He could also add problems involving one or two-digit addends without or with regrouping. He could also subtract problems involving regrouping. There were also a slight improvement in his ability to handle mental computations and word problems.

On the PMDC second grade test in the fall, C could count from 6 to 15, 35 to 46 and could count back from 6 to 1. He could not count by tens from 10 to 130, or from 26 to 126. When asked to determine the numeral represented by 3 bundles of ten straws and 7 single straws, C counted by ones and gave ten as his answer. He could not count by tens and ones to determine the value of a display of objects or to form a set representing a particular number. C could not use counters to solve an addition or subtraction problem and did not attempt to solve any missing addend problems. C answered 2 of 3 items correctly dealing with concepts of "more" and "less" but was unable to order four numbers (2, 3, 5, 9) from smallest to largest. C did not attempt to answer any questions which called for identifying names for the same number. (6 + 3 and 5 + 4; 4 + 1 and 3 + 2; 6 - 1 and 3 + 2; 5 - 2 and 4 - 1; 10 - 5 and 7 - 2; 4 + 1 and 7 - 2).

During the spring of the second grade, C was given a PMDC retest. At this time, he was able to count back from 44 to 25, count by tens from 10 to 130, determine the number of a set represented by 3 bundles of ten straws and 7 single straws, count by tens and ones to determine the value of a display of objects or form a set representing a particular number. C could also use counters to solve an addition or subtraction problem and solved 2 out of 4 missing addend problems. C answered all items correctly dealing with the concepts of "more" and "less" but still had difficulty ordering four numbers (2, 3, 5, and 9) from smallest to largest. C attempted and worked successfully all questions which were related to identifying names for the same number.

As a second grader in the T1 teaching experiment, C was assigned to the embodiment group (U1) using sticks. Initially, C was a behavior problem but eventually became more interested and involved in his work and was less disturbing to the teacher and other pupils. Even though C often engaged in mischievous behavior such as hiding the teacher's pencil, notebook, or teaching materials, there was something likeable about C. He grew fond of the teacher and on several occasions drew pictures during art periods and presented them to the teacher during the mathematics period. Each time the teacher was given a picture, she would thank C and hug him, and he would light up like a Christmas tree.

C worked well on most tasks and often grew tired of stick manipulations. He looked forward to lessons at the symbolic level and would often ask, "Are we going to write today?" Sometimes C would make careless errors even though he knew the concept because he wanted to finish first. He did not like to wait for the teachers to give instructions because he felt that he could figure out what to do without this help. C was often the first one to solve a difficult problem without cues or prompts from the teacher.

Some days C would have difficulty trying to stay awake during class time. It was thought by his homeroom teacher that he stayed up late nights and did not get the proper rest.

The following disucssion of C's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

Counting and forming displays with sticks to represent numbers less than 100 was not a difficult task for C. He would make occasional careless errors but was usually able to correct his mistakes without prompts or cues from the teacher. Although C could count stick displays at the onset of the teaching experiment, he had difficulties reading terms such as one, ones, ten, tens bundle, bundles, stick, sticks, and filling in the blanks on worksheets containing exercises such as:

4 bundles and 2 sticks

____ tens and ____ ones

____ and ____

____.

C could usually fill in the second line correctly but had trouble determining what should go in the blanks in the third line (___ and ___). It was some time before he realized that he needed to fill in the value of his tens and the value of his ones. Eventually C overcame this difficulty and had no trouble filling in similar worksheet items at a later date.

C could tell how many sticks he had in all for a display of 1-9 bundles and could show stick displays correctly for numbers from 10-90 when given to him orally by the teacher.

C would get sleepy and restless on occasion and often resorted to attention-seeking behavior. When asked to write the numeral for 65 on a particular day, C wrote the numeral correctly using his left hand to write the 6 and using his right hand to write the 5. While other children in the group would reverse digits when writing two-digit numerals, C never did this.

Given a two-digit numeral, C could explain the meaning of the digits verbally in terms of tens and ones and could use sticks to explain the numerals. On worksheets, he could write and tell the number of tens and ones that were in a two-digit numeral but had difficulty determining the number of ones.

In the spring of the year, C made a smooth transition to numbers between 100 and 200 represented by three-digit numerals. C could count from 1-100 by ones independently. C was a good thinker and would often demonstrate this ability by independent observations that he made. When shown a picture of 10 groups of 10 sticks, one boy in the group remarked, "That looks like 100 sticks." C followed with the statement, "That's the same as a bundle of bundles."

C had difficulties determining how many tens were in a numeral such as 150 or in a picture stick display representing a three-digit numeral. His response would often denote the number of ones rather than the number of tens. Samples of written work further illustrate these difficulties.

Picture Display	C's Response
1 bundle of bundles	100 tens
1 bundle of bundles and 5 bundles	150 tens
2 bundles of bundles	200 tens

Translating a numeral such as 150 from expanded form to ___ tens was difficult for C and he never seemed sure of his response. If there was any indication that he was wrong, he would correct his mistake. It was as if he had 2 answers available and if the first answer wasn't correct, then it would have to be the second one. Samples of written work indicate these insecurities:

"6 tens, 1 hundred = 61 tens," later changed to 16 tens when he discovered that his first answer was incorrect.

"0 tens, 2 hundreds = 2 tens," later changed to 20 tens when he discovered that his first answer was incorrect.

C could count stick and picture displays for numbers greater than 200 and count form stick displays to represent given numbers. Generally, he had no difficulties in determining the number of hundreds, tens, and ones contained in a three-digit numeral. Sometimes, C would make careless errors such as:

140 = _1_ hundred, _0_ tens, _4_ ones

C had difficulties reading three-digit numerals and rarely made mistakes. C also had the ability to change a given stick display so that it would show the same number but with "more tens" or "more hundreds."

At the end of the school year during the final evaluation interview, C was able to transfer his concept of place value to a new situation in that he was able to represent 2 three-digit addends correctly with beans, a manipulative that was not used in the teaching experiment. Other novel situations proved too difficult for C such as:

> Determining the number represented by 2 hundreds, 13 tens. 4 ones; (C said the number represented was "234")
>
> Recognizing that 245 and 1 hundred, 13 tens, 15 ones were names for the same number; (C thought correct answer had to be "2 hundreds, 4 tens, 5 ones).

When shown 156 and a picture display of 156 objects arranged as 14 sets of 10 and 16 ones, C was unable to relate the numeral to picture and did not think the picture went with the numeral. He was also able to represent several three-digit numerals correctly using sticks, a familiar manipulative.

ORDERING

Ordering pictures representing numbers from 10 to 20 was easy for C. He was the first in the group to order these pictures correctly. C found the picture for 10 first, then searched for the others in order, 11, 12, 13, . . . , 29, 30. He looked puzzled when he was told that the picture representing 20 was left but was the first to discover that it belonged in front of 21. C had no trouble ordering the 40-50 sequence but had trouble ordering the 80-90 sequence. When given a set of pictures to order representing the numbers for 23, 43, 34, 75, 90 and 70, C asked for numbers that he thought had been omitted such as 24, 25, etc. However, when told to order what he had, he was able to order the numbers from smallest to largest. When presented a set of pictures depicting numbers between 110 and 200, C successfully ordered them. He did not search for a pattern but simply evaluated each picture.

C had difficulty reading the phrases, "is less than" and "is greater than" and also the corresponding symbols ($<$, $>$). He had similar difficulties constructing number sentences of the form:

___ "is greater than" ___

___ "is less than" ___

___ $>$ ___

___ $<$ ___

C frequently knew which of two numbers was the greater from a picture or stick display but had difficulties verbalizing or writing formal statements to this effect. In time, C overcame these difficulties and learned to deal with the formal statements in both written and verbal form. After appearing to understand concepts dealing with "more" and "less" in comparing two numbers, C had trouble determining whether a particular sentence was true or false. For example, he knew that "21 was less than 35" but couldn't tell whether the sentence, "21 is less than 35" was true or false. The words, "true" or "false" did not have any meaning for C. Yet, if he was asked whether the sentence was "right" or "wrong," he could frequently give the correct response.

After representing a particular number with a stick display, C could generally show 1 more, 10 more, 100 more, 1 less, 10 less or 100 less. C was unable to transfer this same knowledge to picture sets and often chose the incorrect picture when asked to show a picture that represented 10 more or 10 less than a given picture. For instance, C showed the picture for 343 sticks when asked to find a picture depicting 10 more than the picture representing 243 sticks. Selecting pictures to show 1 more or 100 more than a given picture seemed easier than selecting the picture to show 10 more than a given picture.

Similar difficulties were observed at the symbolic level when C had to show 1 more, 1 less, 10 more, 10 less, 100 more, or 100 less than a given number. Again, the concepts of 10 more and 10 less appeared to be more difficult than the concepts of 1 more, 1 less, 100 more or 100 less.

During his final evaluation interview, C's concept of ordering enabled him to determine that 10 less than 253 was 243. When asked why 243 was correct, he replied, "You take away 10." When asked what number was 10 more than 137, C replied, "127." When asked why this was so, he replied, "You add 1 more ten." Apparently, C knew what to do but did not realize that he had taken away a ten instead of adding a ten. C was also able to represent a three-digit numeral using a stick display and then change the display to show more tens. He also recognized the fact that he could regroup his sticks to show more tens but that the value of the number represented by the original stick display stayed the same.

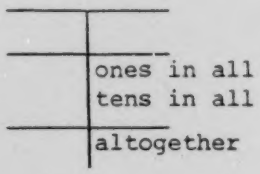
ADDITION AND SUBTRACTION WITHOUT REGROUPING

C had no real difficulties showing the addition and subtraction of number pairs; however, he would make occasional errors due to errors in counting picture or stick displays. C could select sequential pictures for a given addition problem and put them in order. He could look at a picture of a stick display representing an addition problem and could show the answer with sticks.

In the beginning, C had problems recording addition problems and would sometimes put the numerals representing ones under numerals representing tens and vice versa. Samples of his recording difficulties are noted below:

$$\begin{array}{r} 24 \\ +2 \\ \hline 26 \end{array} \qquad \begin{array}{r} 45 \\ +3 \\ \hline 48 \end{array}$$

On worksheet exercises employing the following place value chart format, C had trouble filling in the blanks for the intermediate step. (Ones in all and tens in all).



When C solved subtraction problems he would frequently use double zeros (zeros in the one and tens place) even though he had been taught that the use of one zero was sufficient. For example, he would write:

$$\begin{array}{r} 85 \\ -\,85 \\ \hline 00 \end{array}$$

On worksheets that contained both addition and subtraction problems, C would sometimes overlook the operation sign and add when he should have been subtracting. An example of this error appears in the following samples of his work:

$$\begin{array}{r} 60 \\ -\,30 \\ \hline 90 \end{array} \qquad \begin{array}{r} 85 \\ -\,2 \\ \hline 87 \end{array}$$

When addition and subtraction was first introduced, C did not have a command of many basic facts. He quickly learned strategies for both addition and subtraction and readily used them when deemed necessary.

Prior to being taught how to regroup, if C had problems such as

$$\begin{array}{r} 14 \\ -\,5 \\ \hline \end{array} \qquad \begin{array}{r} 16 \\ -\,7 \\ \hline \end{array}$$

he would not attempt to use subtraction strategies. Instead C would subtract the smaller numeral from the larger numeral in the ones column and bring down the numeral in the tens column. When asked to explain his answer of 11 for 14 - 5, C said, "5 take away 4 = 1; bring down 1." A similar explanation was given for 16 - 7 and other problems of that type.

After much practice, C became proficient with most of the basic facts for addition and subtraction. On a subtraction achievement test, C scored very well.

ADDITION AND SUBTRACTION WITH REGROUPING

C had some problems adjusting to both addition and subtraction problems with regrouping. When he was first introduced to the addition form board, C had difficulty displaying the correct number of sticks on the "tens in all" line. He wanted to count the ten he got as a result of trading as a part of the tens in all line and would have 1 ten more than what he should have in his sum. This difficulty was carried over to the iconic and symbolic levels for awhile. When given a picture stick display representing 38 + 57, C solved it as follows:

```
 38
+57
 15   ones in all
 90   tens in all
---
 95   (correct answer was obtained by
      counting picture stick display)
```

Other errors that C made were due to careless errors in basic facts. For example, he solved 58 + 28 as follows:

```
 58
+28
---
 11
 70
---
 81
```

For a while, C had problems learning to trade in preparation for regrouping in subtraction. Some of his initial trading errors are indicated below

0	2	3	10	5	10	1	11
1	0	3	3	6	8	2	0

Trading errors were carried over to subtraction problems for a while as revealed by the following sample of his work:

```
 2 17
 30
-17
---
 10
```

Eventually C learned how to trade but had difficulties when a zero was in the ones place in the minuend. He treated a problem such as "zero take away nine" as though in were "nine take away zero." C's difficulties with zero are illustrated by the following examples:

60	50	20	30
-39	-27	-8	-15
39	37	28	25

Sometimes C would subtract the numeral representing the smaller number of ones from the numeral representing the larger number of ones instead of trading. Examples of this error appear below:

48	17
-39	-9
61	12

When working subtraction problems, C preferred to put in an extra step even though he was not taught to do it this way for the algorithm. C's method is illustrated below:

29
- 7

2
20

22

During the final evaluation interview, C was unable to transfer his concept of regrouping to a new situation, that is, to the addition and subtraction of three-digit numerals in 6 out of 7 problems.

Using sticks for the first three items and pencil-paper for the last two items, C's incorrect answers are noted below:

Sticks

345	156	524
+173	+178	-183
408	223	241

Pencil-Paper

426	237
+182	-176
518	0611

C was, however, able to use his knowledge of regrouping to solve the following problem correctly with beans, a manipulative he had not been taught to use.

245
-148

97

C's procedure was as follows:

	H	T	O
represented 245 with beans	2	4	5
traded 1 hundred for 10 tens and showed	1	14	5
removed 1 hundred		14	5
traded 1 ten for 10 ones and showed		13	15
removed 4 tens and 8 ones			
showed for final answer		9	7

III. THE CASE OF D

by

Ella Barco

BIOGRAPHICAL AND TESTING INFORMATION

D is a female child born February 2, 1968. She lives in a house shared by another family. D and four other children, a seven year old brother, a four year old sister, a one year old sister, and a four month old sister are reared by their mother. Although there is no father living in the home, D's mother provides her with love and affection and appears genuinely interested in her welfare.

During the school year, there was evidence of a lot of commotion in the home and D seemed to seek out situations which afforded her adult protection. Whenever she went on the playground, she would always stay close by her teacher and would never join in games with other children. D has very little self-confidence.

D's mother was pregnant for a portion of the school year and did not work outside the home. Her family would be classified in the lower socioeconomic level based on annual income.

D, a very soft spoken child, was characterized by her teacher as having a totally passive personality. Other children found it an easy job to walk over her. D never tried to form friendships with other children and had no real friends. However, she was not disliked by other students in the room. In fact, one of the smartest children in the room chose D to be her partner on a field trip and also selected her as a friend on a classroom sociogram.

D attended kindergarten, first, and second grade at the same elementary school. She was a poor speller and reader and was a participant in the ESEA Title 1 program which provided opportunities for needed individualized instruction. D eventually received help at home, and her reading and spelling ability improved.

D was an insecure child and never wanted to try anything new unless it was a game. She thrived on positive reinforcement and praise. D would sit passively ly in the classroom and never say anything unless persuaded by the teacher. She never wanted to go to the board or do anything individually that would focus attention on her.

D was not a good student in the first grade. She scored below grade level on the Comprehensive Test of Basic Skills (CTBS) administered at grade 1.6 with grade equivalent scores of .6, .1, and .1 in reading, language, and mathematics, respectively.

During second grade, D was still functioning below grade level; however, there was a noticeable improvement in her mathematics grade equivalent score on the second grade CTBS. Her grade equivalent scores for this test were 1.2, 0.6, and 1.4 for reading, language, and mathematics, respectively. Notice the 1.3 grade equivalent increase in her mathematics test score. Her entering second grade Otis-Lennon IQ was 73. Although operating below grade level, D tried hard to complete all tasks even though she worked slowly and encountered numerous difficulties along the way.

D had an entering KeyMath grade equivalent of 0.7. She could compare the numerousness of sets of nine or fewer objects but could not identify missing numbers in a written sequence (___, 6, 7, ___, 9; ___, 19; 98, 99, ___, 101). D could add two one-digit addends involving sums less than ten but could not add problems involving sums greater than 10 without or with regrouping. Neither could she solve missing addend problems. D could not subtract problems involving minuends greater than five and subtrahends greater than two. D could not perform successfully on any mental computation problems. She could only solve one word problems which involved the taking away of one object from a set of two objects. Having or not having a picture prompt did not seem to make a difference in D's problems solving ability.

In May of the second grade, D was again administered the KeyMath Test and received a grade equivalent of 2.1. At this time, D was able to identify the missing number in sequence (___, 19). She was still unable to add problems involving sums greater than 9 without or with regrouping. D was able to subtract problems such as 5 - 3; 8 - 2; 16 - 12; 14 - 6. She was still unable to subtract problems involving regrouping. D showed some improvement, however, on solving word problems presented with a picture stimulus.

On the PMDC second grade test in the fall, D had numerous counting difficulties. She could only count from 35 to 39 when asked to count from 35 to 46. D could not count back from 6 to 1 or from 44 to 25. She could not count forward by tens from 10 to 130 or from 26 to 126. When asked to determine the number of a set represented by 6 bundles of ten straws, D unsuccessfully counted by ones. She could not count by tens and ones to determine the value of a display of objects or to form a set representing a particular number. D could not use counters to solve an addition or subtraction problem and had no success with missing addend problems. D could tell which number was more or less from sets of two elements ({7, 4}, {8, 12}, and {19, 31}); however, she was unable to order four numbers (2, 3, 5, and 9) from smallest to largest. D also experienced failure on all tasks requiring the identification of names for the same number such as: 6 + 3 and 5 + 4; 4 + 1 and 3 + 2; 5 - 2 and 4 - 1; 4 + 1 and 7 - 2.

During the spring of the second grade, D was given a PMDC retest. This time she was able to count from 6 to 15, count by tens from 10 to 130, count by tens and ones, determine the value of a display of objects, and form a set representing a particular number. She still could not count by tens from 26 to 126, solve missing addend problems, or identify names for the same number.

As a second grader in the T1 teaching experiment, D was assigned to the teaching group (U1) using sticks. Of the six children who were originally assigned to this group, D seemingly had the most difficulties with tasks at the manipulative, pictorial, and symbolic levels. She was usually the last one to finish assignments and frequently made the greatest number of errors. For some topics, D worked successfully at the manipulative and pictorial levels but experienced a great deal of difficulty at the symbolic level. D took longer to develop concepts and seemingly experienced retention difficulties within short time spans.

The teaching of a new concept or procedure to D sometimes blotted out a previously learned concept or procedure.

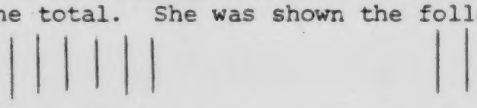
When the experiment first began, D lacked self-confidence and would copy what she saw the others doing; however, as time progressed, she gained more self-confidence and showed more independence in performing tasks.

The following discussion of D's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

Counting and forming displays with sticks to represent numbers less than 100 was a difficult task for D for a long while; however, she slowly overcame some of these difficulties. Initially, counting difficulties were due to D's inability to do rational counting. If she were counting a stick display, she would touch some of the sticks and skip over other sticks and would not realize that she had not accounted for all sticks in the display. D would frequently lose her place in the counting sequence and have to start over. She had trouble recalling the last number she had counted before adding on to a stick display and would usually have to recount the entire display. D eventually learned how to do rational counting.

D also had problems related to the counting of 2 displays of sticks to obtain the total. She was shown the following stick display:



D counted each set separately (6, 2) and said that the total number of sticks was 62. For a similar display consisting of 2 sticks and 7 sticks, she said that the total number of sticks was 27. For D, the total number of sticks in two different sets appeared to be the number formed using the numerals assigned to the total in each set. When shown a display of 14 sticks as 1 bundle and 4 sticks, D was unable to count the sticks or write the answer. She knew that there were ten sticks in a bundle but could not count by tens and ones. D would either count by ones ("one, two, three, four . . ."), or she would count

by tens ("ten, twenty, thirty, forty, fifty . . ."). It took D a considerable period of time to learn to count by tens and ones, especially, for the numbers between 10 and 20. She was very proud of her accomplishment the day that she learned to count from 10 to 20 independently.

When writing a two-digit numeral, D would frequently reverse the order of the digits. For other numerals, what D wrote had no relationship to the numerals in the number being represented. Sometimes D would count a stick or picture display correctly and give a correct oral answer for the total number of sticks but would be unable to transfer the oral answer to the corresponding written symbols.

When asked to make stick displays for particular numerals, D's stick displays were frequently incorrect. For example, when asked to show a stick display to represent 38, D exhibited 7 bundles. This would suggest that D did not interpret 38 as 3 tens and 8 ones.

D never appeared to be thinking about how she would solve a particular problem. It was as though, "I'm expected to give an answer, so I must 'say' something or 'do' something." Rarely, if ever, did D say, "I don't know how to do that" or "I don't know the answer to that question."

Filling in the blanks on worksheets also proved to be a difficult and sometimes laborious task for D. For a long while, the words tens, ones, bundles, and sticks were words that were read but had no relationship to the numerals used in representing a particular number. For a picture display showing four bundles and 4 sticks, D filled in a worksheet item as follows:

40 bundles and 4 ones

4 tens and 4 ones

40 and 4 ones

44.

In filling in blanks in response to a number read by the teacher, D would have difficulties similar to the ones indicated by the following sample of her work.

Number read:	D's written response:
42	4 tens and 2 ones
	24 and 2
	42

D arrived at the correct answer by recounting the picture display.

At first D had trouble determining how many tens and how many ones were in a two digit numeral and had difficulty equating 1 ten with 10. Eventually, D overcame this difficulty, however, she never really felt sure of herself when asked to determine how many ones were contained in a particular two-digit num-

eral.

In the spring of the year, D encountered additional difficulties in trying to adapt to three-digit numerals between 100 and 200. She had problems counting and forming stick displays to represent numbers between 100 and 200. When shown a picture of 100 sticks and asked to count orally in a group situation, D did not know what number followed 99 in the counting sequence. When counting stick displays of bundles representing numbers which were multiples of ten between 100 and 200, D would not know what to say after reaching 100 and would usually start the counting sequence over again. For example, D counted 15 bundles in the following manner:

"10, 20, 30, . . . , 100, 10, 20, 30, 40, 50."

When given a stick or picture display containing bundles of bundles, bundles, or sticks, D had difficulty determining the total value of the display as well as the number of tens and the number of ones in the entire display. When shown a stick display of 10 bundles and 1 stick and asked to tell how many sticks in all, D responded, "110." D's difficulties were carried over to her written work as indicated by the follwoing samples of her written work.

Teacher's Display		D's written responses:
15 bundles	=	100 tens
10 bundles	=	90 tens
19 bundles	=	18 tens = 109
20 tens	=	200 tens

Other worksheet items answered incorrectly by D were:

160	=	61 tens
130	=	0 hundred, 3 tens
19 tens	=	1 ten, 7 hundred

Translating a numeral from expanded form to ___ tens was an equally difficult task for D as revealed by the following responses:

6 tens, 1 hundred	=	20 tens
0 tens, 2 hundred	=	0 tens
2 hundred, 0 tens	=	20 tens

Numbers greater than 200 had an even more devastating effect on D's performance. Counting, interpreting pictures of stick displays, and representing numbers with stick displays was very frustrating at times for D. Samples of her difficulties were:

Numeral to be displayed with sticks:	D's Response
271	"241"
Stick display given:	**D's Response:**
2 bundles of bundles and 4 sticks	"240" (counted: "100, 200, 210, . . . , 240")
Picture Displayed:	**D's Response:**
2 bundles of bundles and 6 bundles	"80" (counted: "10, 20, 30, . . . , 80")

D frequently counts bundles of bundles and ones as "tens" and tens as "ones." D never learned how to read three-digit numerals independently. When asked to read three-digit numerals, she would respond by telling the number of hundreds, tens, and ones she thought the number contained.

At the end of the school year, D's concept of place value left much to be desired; and this was borne out in her final evaluation interview. Place value difficulties manifested during the interview were similar to the ones D had encountered throughout the school year such as:

(a) Inability to distinguish between tens and ones, and hundreds and tens on occasion when reading and writing numerals:

read 156 as: "1 hundred, 5 ones, 6 ones
read 524 as: "5 tens, 2 ones, 4 ones"
read 173 as: "17 + 3"

(b) Inability to determine the number of tens in a given number:

E: "How many tens are in 156?"
D: "14 tens"

(c) Inability to relate a number and a picture representation of the number:

didn't think 156 was depicted by a picture display of 156 objects

(d) Inability to show correct stick displays to represent a number:

represented 148 as: 1 ten, 4 ones, 8 ones
represented 183 as: 1 ten, 8 ones, 3 ones

Lastly, the final evaluation interview showed that D was unprepared for transferring place value concepts to new situations. When shown the following problem, D was unable to determine the number it represented:

H	T	O
2	13	4

When shown 245 = 1 hundred, 13 tens, 15 ones, D did not think that this was correct but was unable to tell why she thought it was not correct.

ORDERING

D did not know what to do when told to order pictures of manipulatives representing numbers from 10-20. She watched to see what her classmates were doing and simply imitated their behavior. After improving her counting ability, she was then able to order the pictures without prompting. D had considerable difficulty ordering picutres representing numbers from 20-30. She could not keep track of the numerical values of particular pictures and recounted the same picture several times. D did better with the 40-50 sequence. It appeared that for this picture set, she ordered the pictures on the basis of the number of ones rather than determining the value of each picture as she had done previously. Ordering difficulties seemed to have returned when D attempted to order the 80-90 sequence. Again, she attempted to determine the value of each picture and was unable to remember what she had counted and had to recount numerous times. D never learned how to order the larger numbers such as 110-200 and copied what others were doing.

D could not determine which of two sticks or picture displays was "more, less, greater, or fewer." If the two displays each involved bundles and sticks, she would pick the one that contained the most bundles to represent the greater number. If, however, the number of bundles in each display was the same, D was unable to look at the ones and determine which display was "more" or "less." D had difficulty reading phrases and symbols for "is greater than" ($>$) and "is less than" ($<$). She had similar difficulties constructing number sentences of the form:

___ "is greater than" ___

___ "is less than" ___

—— $>$ ——

—— $<$ ——.

D had no idea when to use the "is greater than" or "is less than" symbols ($<$, $>$) and did not know how to read sentences such as:

$3 < 5$

$5 > 3.$

D never knew when a sentence containing the "is greater than" or "is less than" phrases or symbols was true or false. The terms (true, false) had no meaning

for D. Her answer was either a guess or the repetition of an answer given by someone else.

After representing numbers with stick displays, D had trouble deciding what she would do to show 1 more, 1 less, 10 more, 10 less, 100 more or 100 less. She finally got to the point where she knew that she must add additional sticks to show "more" and take away sticks to show "less" but she frequently had trouble deciding which sticks to add or take away. Should it be ones, tens, or hundreds? D never seemed sure and often chose the wrong sticks. For example, when asked to show 1 more than 53, D added an additional bundle to her display of 53 sticks.

During her final evaluation interview, D's concept of ordering did not enable her to determine the number that was 10 more than 137 or 10 less than 253. When asked, "What number is 10 more than 137?" D's reply was "700." When asked, "What number is 10 less than 253?" D's reply was "500." D's concept of ordering left much to be desired.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

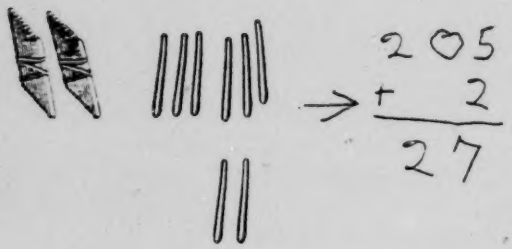
D had trouble with addition and subtraction problems without regrouping at the manipulative, pictorial, and symbolic levels.

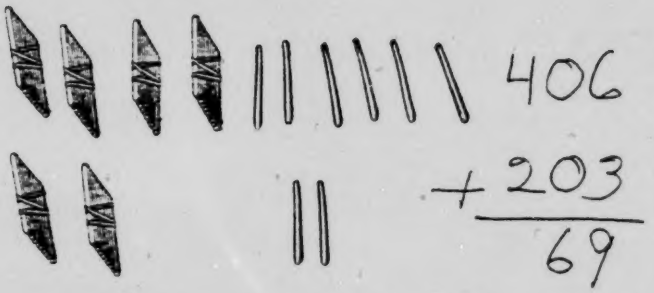
At the manipulative level, D did not always represent the addends correctly when working addition problems. Similar difficulties were encountered in representing the minuend and subtrahend in subtraction problems. On occasions, D would correctly set up problems but would make numerous counting errors which led to incorrect answers.

When using a form board to work addition problems, D would frequently join her tens first. For a long while, D was dependent on the form board and did not want to attempt any addition or subtraction problems at the symbolic level without it, but eventually developed a bit of confidence and attempted to work without it. Eventually D got to the place that she did not want to use the form board, especially when the other members of the group were not using theirs, but she made so many mistakes without it, she was encouraged to return to the use of the form board.

When working with pictures, D had trouble selecting sequential pictures for a given addition problem and putting them in order to show addition. D would select pictures based on the total number of tens and not realize that the ones were incorrect. D also had trouble translating addition picture problems into their corresponding written symbols. Samples of her translations from pictures to symbols for addition problems appear below:

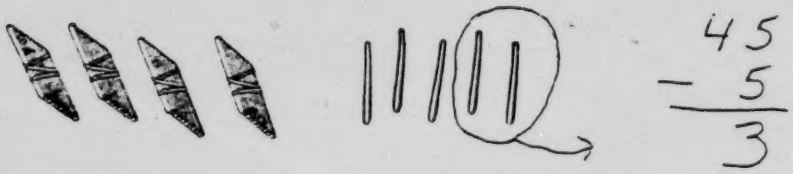
Picture	D's translation
	5 3 + —— 8





In each of the above samples, D has represented the problems incorrectly symbolically but her answer seems to reflect that she had some notion of the concept of addition in that she used the pictures the determine the total number of sticks.

Similar difficulties were encountered by D in translating pictures for subtraction into written symbols as noted in the following samples of her work.



Unlike addition, the above samples of D's work do not reflect an understanding of the concept of subtraction. This was borne out at the symbolic only level in D's response to problems such as:

8	50	13
-3	- 2	- 4
8	50	13

In these and other subtraction problems, D simply brought down the minuend for the answer instead of subtracting the subtrahend from the minuend. In working subtraction problems involving two-digit minuends, it was not uncommon for D to subtract the ones and forget to subtract the tens or vice versa. This was borne out in samples of her work below:

88	66	93
- 7	-23	31
1	4	6

D also had trouble trying to decide where to record her ones and tens in using the place value chart format for both addition and subtraction problems as noted below:

Picture

D's Translation and Recording

9	5
	5
	9
9	0
9	5

In the previous example, D's answer was correct even though she had a faulty algorithm. D had learned to count some picture displays correctly and knew that the algorithmic answer corresponded to the total value of the stick picture display. D also had recording problems when she was not using the place value chart format. Often times, she would place ones under tens and tens under ones as illustrated by samples of her work below:

$$\begin{array}{r} 60 \\ -\ 40 \\ \hline 20 \end{array}$$

(placing tens under ones)

$$\begin{array}{r} 13 \\ -\ 4 \\ \hline 13 \end{array}$$

(placing ones under tens)

For numbers less than 100, as long as D was given only addition problems or only subtraction problems, she eventually learned how to translate the stick or picture display into written symbols. Whenever the operations of addition and subtraction were combined on a single worksheet, D had difficulties distinguishing between addition problems and subtraction problems at both pictorial and symbolic levels. Even when the operation signs were written beside the problem, D still continued to confuse addition and subtraction problems.

D was weak in basic facts and was taught strategies to use for finding needed facts in addition and subtraction problems. As long as D focused on a single strategy, she did all right; but as soon as she had to switch from an addition strategy to a subtraction strategy, she became thoroughly confused and made numerous errors.

With the passage of time, D improved on translating stick displays and picture problems for addition and subtraction problems involving two-digit numerals into symbolic form. However, place value concepts, knowledge of basic facts, algorithmic and recording problems continued to follow D throughout the T1 teaching experiment. She never knew which strategies to use for addition or which

ones to use for subtraction.

D wanted to continue the use of her form board for addition and subtraction problems involving regrouping at the symbolic level because she felt insecure without it. Whenever D was encouraged to work without her form board, she did so, but her work seemed laborious and tedious and numerous errors were made.

In subtraction with regrouping, D was plagued with "trading" problems and spent a considerable amount of time trying to learn how to determine when trading was necessary. Eventually, D understood that she was supposed to trade whenever she didn't have enough ones but she did not always apply this rule and would frequently subtract the smaller number of ones from the larger number of ones or add the ones together as indicated by her work below:

$$\begin{array}{r} 17 \\ -\ 9 \\ \hline 12 \end{array} \qquad \begin{array}{r} 34 \\ -25 \\ \hline 19 \end{array}$$

ADDITION AND SUBTRACTION WITH REGROUPING

D had numerous difficulties with addition and subtraction problems involving regrouping at all three levels--manipulative, pictorial, and symbolic. D did not appear to ever completely understand ideas of regrouping and renaming. She did not seem to understand that a number could be represented in more than one way. When asked if 12 ones were the same as 1 bundle and 2 sticks, she replied, "No." D also had trouble answering the question,"Are there less than 10 ones?" When the question was rephrased,"Do you have enough to make a bundle?" she would be more inclined to answer correctly.

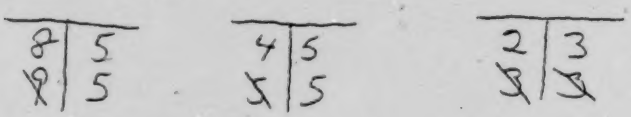
Some of D's errors in addition and subtraction involving regrouping at the manipulative level were due to her inability to count accurately and represent numbers correctly. D also had difficulty at the iconic level showing addition problems with sticks based on a given picture display and finding pictures corresponding to selected oral statements of an addition problem. D was never able to explain the regrouping process without prompting from the teacher. In fact, she did very little verbalizing unless asked a direct question.

When working at the symbolic level with pictures, D had difficulty with worksheet problems such as the following:

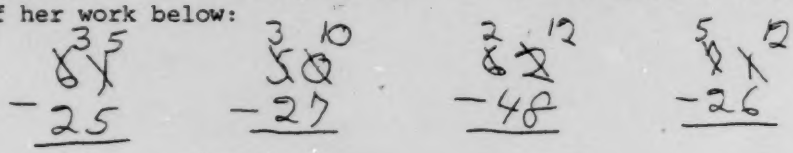
5	8	
+ 1	6	
1	4	ones in all
7	0	tens in all
8	4	altogether

27

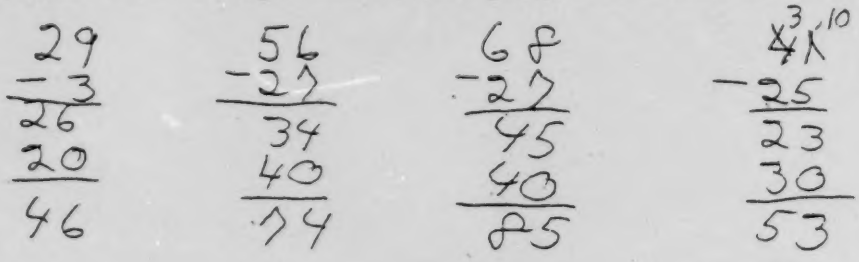
The intermediate step ("ones in all"; "tens in all") gave D considerable difficulty. She frequently counted the ten that she got as a result of trading twice, that is, once on the "ones in all" line and again on the "tens in all" line. Errors in basic facts continued to bother D on addition and subtraction problems with regrouping. Strategies for finding unknown facts continued to be confusing to D. In preparatory exercises involving the concept of "trading," D's responses on many exercises were as follows:



When D attempted to trade, she did not seem to know what she was doing. It was as though she thought you must rename both the tens digit and the ones digit but no particular rule was to be employed in the remaining process as indicated by examples of her work below:



On worksheets involving both addition and subtraction problems, D had difficulty trying to decide which operation to use and attempted to incorporate both operations into a single subtraction problem as illustrated below:



On written work, D often added or subtracted the ones first which led to other problems when regrouping was involved.

During the final evaluation interview, D was asked to solve three-digit addition and subtraction problems with regrouping. D not only had difficulties representing various numerals with sticks, but counting, place value concepts, regrouping, and recording problems were again manifested. D was unable to apply her concept of regrouping to a new situation, that is, the addition and subtraction of three-digit numerals with regrouping.

IV. THE CASE OF F

by

Stewart Wood

BIOGRAPHICAL AND TESTING INFORMATION

F was born September 21, 1968, the youngest of seven children. One brother is a year older than F; the other siblings are much older and out of school. For a period of her life, F was passed from aunt to aunt; she now lives with her grandmother, who has adopted her.

F has attended the same school in both first and second grade and has been in the Title I reading program both years. In general her communication skills are not good; she began second grade in the lowest group for every subject except math. She is conscientious about her work, likes to be done first, and is upset if she has to redo an assignment.

F is a happy child who likes adult attention. Her classmates regard her as something of a teacher's pet and tend to isolate her from play groups. She is not strongly disliked, however, and always has something to do. F has a creative imagination and a good sense of humor.

Math is F's best subject. During first grade she received special daily small-group instruction as part of a PMDC experimental group. This work centered on addition, subtraction, and equality. (F seemed to thrive in having a "special" math teacher and consistently outperformed the expectations that resulted from her Otis-Lennon IQ of 90.)

In the fall of second grade, F's KeyMath grade equivalent was 1.9. She was able to do one-digit addition and subtraction in both concrete and symbolic forms, but not two-digit problems. She could solve simple "story" problems accompanied by pictures, including a partition-division problem ($8 \div 2 = \square$). Her command of money, time, and measurement skills was at or above grade level.

Im May of her second-grade year, F again was given the KeyMath Test, scoring a grade equivalent of 2.6. F showed growth in counting skills involving the completion of a sequence (__, 6, 7, __, 9 or 98, 99, __, 101) and in addition and subtraction skills--she now could do two-digit problems involving regrouping. F showed a marked improvement in mental computation and numerical reasoning, solving such problems as 5 + 5 - 4 + 7 (given orally and solved mentally) and " $\triangle + 9 = 10$, $\triangle - \triangle = \square$; what number goes in the box?" There was little change in her performance in the practical areas of money and measurement, in which she had received little instruction during the year.

On the PMDC test given at the beginning of grade two, F counted by ones successfully on all tasks except 44 to 25 backwards. She could count by tens to 100. F was able to count and to construct sets of straws (bundles of ten and single straws) for two-digit numbers, but was not able to use colored poker chips (red chips for tens, white for ones) to do the same tasks. She could use beans to illustrate and solve addition problems (2 + 4, 18 + 5) but not subtraction (7 - 3, 23 - 7). While she correctly identified 12 as more

than 8, she also identified 7 as less than 4 and 19 as more than 31.

In the spring, F still had a few difficulties with the counting tasks: She omitted 40 and 30 in counting from 44 to 25; she needed a prompt to count by tens from 26 to 106. She succeeded on all the place value tasks, using both chips and straws, and she performed additions and subtractions with ease using beans. On ordering tasks, she erred only in identifying 19 as more than 31.

On the Comprehensive Test of Basic Skills, administered at grade 2.6, F scored an overall grade equivalent of 2.3, with reading scales in the 18-25 percentile range (2.0), language scales in the 17-63 percentile range (2.3), and math scales in the 59-86 percentile range (3.1).

During her year in second grade, F was assigned to the group (U2) using Dienes blocks. Overall, she responded well to mathematics instruction; she was anxious to please, to work quickly and accurately. She particularly liked "writing work" as opposed to orally-directed work with Dienes blocks or with pictures, possibly because the "writing work" involved a product which she could proudly save. She grasped new concepts easily at all levels of presentation - enactive, iconic, and symbolic - and had little difficulty verbalizing or demonstrating with blocks the symbolic processes she learned. She and another child in the group were consistently the first to finish individual work and the most active contributors to group work, although of these two children, F was the more patient with slower workers and often would "help" them by asking teacher-like leading questions.

PLACE VALUE

From the beginning, F had no trouble forming or counting displays or pictures representing two-digit numbers. If a display had one or two longs and less than five units, she often responded immediately with the total. When she needed to count, she counted by tens and ones (10, 20, 30, 31, 32, . . . , 36) or sometimes did the tens immediately and counted on (30, 31, . . . , 36).

Initially, F made errors in written work of the form

_2_ longs and _3_ ones

_20_ tens and _3_ ones

_2_ and _3_ ones

_23_

done either from a block display or picture. A discussion of "tens" clarifying that she could indicate tens either by using the word or by using a zero (placeholder) helped, although she continued to make occasional errors, expecially with numbers involving 1 ten:

_10_ tens + _0_ ones

_10_ + _0_

_10_.

In other written work she was quite successful, including completing expanded numerals presented in random order without blocks or pictures (_1_ ten and _6_ ones . 16, 90 _9_ tens).

Of interest is the fact that after two weeks work, when she was first asked, "How much is 2 longs and 3 units?" without either blocks or a picture present, she couldn't answer. Thus her oral and written work with manipulatives didn't automatically prepare her for work without aids, although she was a quick learner in either situation.

At her November interview, F was quick to describe a number in three different ways. (4 tens + 6 ones, 40 + 6, 46). She chose an abacus (which she had not used before) to represent 23, flipping 20 beads on one rod and 3 on another. However, in showing 32 on the abacus she flipped 20 beads (saying "20"), flipped another 20 (saying "30") and then flipped 2 more (saying "32").

Throughout these first weeks of work, F was able easily to explain verbally or with blocks the meaning of each digit in a two-digit number. Her errors, even in an unfamiliar situation, seemed to stem from rushing or from "not thinking." In most cases she was able to correct an error on her own when it was brought to her attention.

In the spring of the year, F easily embraced work with three-digit numbers. She readily identified 17 tens as 170 and vice versa. Given a picture of 125 in one form (1 flat, 2 longs, 5 units), she could describe a different picture of 125 (10 longs, 1 long, 15 units) accurately by parts ("100 . . . 110 . . . and 15 ones makes 125") without counting. When asked to trade a display of 1 flat and 2 longs to show 120 another way, she tired of always trading a flat for 10 longs and began to do other trades: 1 flat and 20 units, or 1 flat 1 long and 10 units. She had no trouble completing blanks of the form:

_17_ tens

_1_ hundred, _7_ tens

_170_

given a block, picture, or written stimulus for any one of the three lines.

F could make an accurate block display for numbers like 654 and describe it in terms of hundreds, tens, and ones. She had difficulty describing it as "65 tens and 4 ones" without first doing or seeing the trade of 6 flats for 60 longs. On the other hand, she could count by tens with the blocks (376, 386, 396, . . .), adding a long to the display with each number, and could bridge the hundreds change (to 406) without actually seeing or doing the trade to 4 flats. She had considerable difficulty doing the same kind of counting by tens when working symbolically:

280	209	
197	27	
309	319	320

Counting by ones or by hundreds was easier for F, both with blocks and without. When using blocks, she would anticipate the next number orally; when writing without blocks, she occasionally bridged correctly into thousands:

780	880	980	1080
709	809	909	109

In her final interview, F displayed three-digit numbers using both blocks and an unfamiliar manipulative (beans). While she had some difficulty with nonstandard forms, she displayed considerable facility in her thinking. On one task she was shown the numeral "156" and a picture of 14 ovals, each representing ten, and 16 small squares, each representing one. She was asked if the picture showed the number. This dialogue occurred:

F: No, you don't have enough hundreds.
Question: But there are lots of tens. Maybe there are enough

(F counts 10 tens and stops)

Q: How many more tens:
F: 4, 140
Q: How many ones?
F: 16
Q: So?
F: 150
Q: But I thought you said 140.
F: I traded . . . it's okay.

ORDERING

During her work with place value, F had no trouble sorting as many as eleven pictures to show counting, with one exception. When asked to order pictures representing the numbers 40 to 50, she quickly placed 41 through 49 in order by counting the units in each picture. She then placed the pictures for 40 and 50 to one side and said she didn't know where they went. Only after rote counting from 48 to 52 and from 36 to 42 did she realize where to place the ends of the decade.

In November, prior to receiving specific instruction on the concepts of more and less, F was shown the numeral 20 and asked what number is ten more. She answered, "30, because this number (20) comes before this number (30)." She did similar tasks using beans. Thus her concept of more was already well-formed.

F began work with order relations by making or observing block displays for two two-digit numbers and then making an oral statement of comparison. From the beginning she was equally facile in making sentences using "is greater (more) than" and "is less than." When asked to correct a given false statement, F used a variety of methods: She exchanged the numbers ($17 < 71$ to correct $71 > 17$); she changed one number ($62 < 64$ to correct $62 < 26$); and she changed the re-

lation (23 < 43 to correct 23 > 43). When asked to complete a sentence like "23 is greater than . . . " by showing blocks for the missing number, F mischievously used extreme numbers--including zero and 100.

After the symbols " < " and " > " were introduced, F wrote sentences using " < " whenever possible. She made few errors in using either symbol. In an interview several weeks later F was accurate in identifying the greater or lesser of two numbers and gave articulate reasons; however, she used " > " to mean "is less than" and " < " to mean "is greater than."

In working with three-digit numbers late in the year F relied heavily on block displays, although she rarely needed to actually count the blocks to know the new total. For example, with a display of 87 she was asked what number is 20 more; she added 2 longs to the blocks, said "107," and traded the longs for a flat. For 10 more than 491 she recognized immediately that a trade would be possible and simply replaced the 9 longs with another flat, saying "501." Similarly, for problems like "2 less than 371" she was usually the first child to see that a trade was necessary; she worked quickly and competently.

At her final interview F was asked two oral questions of this type without any symbolic, picture, or manipulative cues. She responded easily.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

When F first began writing addition problems in vertical form, her column alignment was very poor. If she worked a problem like 60 + 3 with blocks, from pictures, or orally, she recorded the sum after observing the display, picture, or saying the answer. Thus columns didn't matter:

$$\begin{array}{r} 10 \\ +\ \ 5 \\ \hline 15 \end{array}$$

However, even after her column alignment improved, she had trouble with problems done without blocks or pictures:

$$\begin{array}{r} 6 \\ +30 \\ \hline 90 \end{array} \qquad \begin{array}{r} 40 \\ +20 \\ \hline 42 \end{array}$$

When asked to read such problems aloud, F immediately recognized her errors and corrected them.

F's command of basic facts was good; when in doubt she added on her fingers quickly or, if blocks or a picture were handy, did a rapid visual count of the total. When the addition form board was introduced, F displayed the

two addends easily. Often she announced the sum immediately, then combined the units and the longs. After two days she rebelled against the formality of the addition board, saying "I hate this . . . I don't need it."

F was the first child in the group to verbalize that "it doesn't matter" whether you add the ones first or the tens. In her own work she was flexible, although she usually recorded the ones first. She had no difficulty at all solving symbolic problems without blocks or pictures, explaining (for 45 + 23), "I added the ones and I added the tens . . . 5 + 3 is 8 and 4 + 2 is 6."

F mastered subtraction with similar ease. She used her fingers more often for basic facts and was less resistant to using blocks and the subtraction form board in her first tasks. When blocks or pictures accompanied worksheets with both addition and subtraction problems, F had no trouble distinguishing; when they were not present, she occasionally confused strategies:

$$\begin{array}{r} 56 \\ +23 \\ \hline 73 \end{array}$$

At her February interview F worked a variety of addition and subtraction problems. She worked quickly, using her fingers rather often. She recognized addition and subtraction problem situations which were described orally ("This bank has 32¢; this bank has 6¢ . . . ") and solved the problems that were posed in her head ("30 . . . 2 . . . 6 . . . 38"). She was able to find and correct errors in given written problems.

ADDITION AND SUBTRACTION WITH REGROUPING

At the February interview, F anticipated work with regrouping. Given the problem 27 + 35, she counted 5 + 7 with her fingers and said "It can't be 12." Then she wrote

$$\begin{array}{r} 27 \\ +35 \\ \hline 512 \end{array}$$

and reached for the blocks. She counted 5, 7, then 12 units, counted 10 of the 12 and said, "This makes a long; that's 60 . . . 62." This occurred prior to any instruction on regrouping. On the subtraction problem 53 - 24 she was not able to break through, even with considerable prompting.

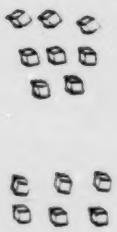
As she began formal instruction in regrouping skills, F worked confidently. With blocks she did not count the total after a trade for "more ones" (e.g., 3 longs, 5 units traded to 2 longs, 15 units) but knew that total was still the same (35). On the other hand, in a sequence of pictures depicting the same trade, she often did need to count ("10, 20, 21, 22, . . . , 35") especially

when asked how many units there were (15). Eventually she latched onto the pattern of "10 more" and could say without counting that the units had gone from 5 to 15.

In adding two-digit numbers, F at first counted the ones-in-all, particularly for sums greater than 13. She used a quick visual count or mental arithmetic for the tens-in-all and found the final total mentally (e.g., for 45 + 38: "8 . . . 9, 10, 11, 12, 13 . . . 70 . . . 83"). As she progressed into written work, she did more and more mentally, using her fingers as needed and ignoring the blocks. She continued to use the long form, consistently combining the ones first:

$$\begin{array}{r} 27 \\ +\ 6 \\ \hline 13 \\ 20 \\ \hline 33 \end{array}$$

Even when she had to write the problem from a picture, worked mentally as soon as she had recorded the addends (rather than from the picture), occasionally making a fact error:



$$\begin{array}{r} 78 \\ +\ 9 \\ \hline 17 \\ 10 \\ \hline 87 \end{array}$$

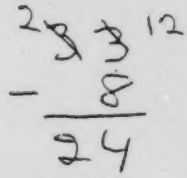
F was the first child in the group to suggest trading in order to work the subtraction 45 - 37. At first she sometimes misjudged when a trade was required (Pictures accompanied these problems):

$$\begin{array}{r} 68 \\ -29 \\ \hline 41 \end{array} \qquad \begin{array}{r} 2\not{3}\overset{16}{\not{6}} \\ -23 \\ \hline 13 \end{array}$$

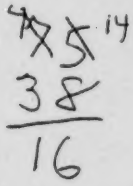
One day she announced that she could tell when to trade "from the ones . . . if it's bigger here (subtrahend)"; however, the next day she traded on every problem, ignoring what she had written for the trade if she discovered she didn't need it:

$$\begin{array}{r} \not{5}\not{0}^{10} \\ -40 \\ \hline 30 \end{array}$$

On her last day of classwork with subtraction, she did a series of problems this way:



When asked to explain one of them, she said "Oops . . . it should be 13." In the interview that followed in April, she worked this problem and explained:



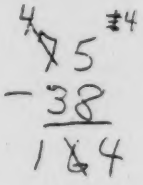
F: I had to take away from the 7 to get 4 so I could subtract. I took away 3 tens from 7. That's (the 4 in 14) for the 4 . . . the 1's not supposed to be there.

(F separates out the 1)

Q: What about the 4 (in 14)?

F: I took one away from the 5.

(F thinks a minute, scratches out the 6 and writes 4):



F: The 4 . . . I took away 4 from the 8.

Q: How did you get the 1?

F: From the 4.

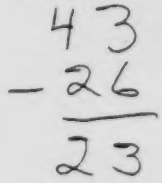
Q: Would you read the problem?

F: 75 take away 38 equals 14.

Q: Is that okay?

F: Yes.

In the same interview F worked 54 - 37 with blocks accurately except that she removed 10 units rather than 7 for the subtrahend. She worked 32 - 8 accurately with beans (cups of 10 and single beans). She was asked if the following problem was correct:



F: 43 take away 26 . . . (works on fingers) . . . equals 23.
Q: Is it right?
F: Yes, 'cause 6 take away 3 . . . no, you have to trade.
Q: What should be the answer?
F: 33.

F had no trouble at all with addition in this interview: using manipulatives, explaining, finding errors. Her gross confusion about subtraction is puzzling. She seems not to have observed the parallels between block manipulations for subtraction with regrouping and the symbolic notation, in spite of several weeks experience with trading activities and the subtraction process. She is not disturbed by bizarre symbolic results, and these examples suggest that she manipulates symbols without much thought to the numbers and qualities they represent. The written subtraction algorithm involves many more steps, as well as strategy decisions, than does the long-form addition algorithm; apparently F could not cope.

In the final interview F solved several three-digit addition problems requiring regrouping using manipulatives and one problem without aids:

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline 5109 \end{array}$$

After several prompts from the interviewer, F was able to correct the sum to 608. With subtraction she also succeeded when manipulatives were provided, making trades accurately as necessary. On the problem without aids she became confused and finally gave an incorrect oral answer.

V. THE CASE OF G

BY

STEWART WOOD

BIOGRAPHICAL AND TESTING INFORMATION

G was born July 9, 1968, the sixth of seven children ranging in age from 20 to 2. His three next older siblings are brothers, ages 9, 11, and 13. He lives with his mother and stepfather, both of whom are employed full time.

G is a very private child; he holds back and is uncomfortable about being touched. He likes to work alone, at a desk by himself rather than at a group table. He does not contribute to class, will not answer questions, and does not like to be singled out -- even for praise. He reads silently and falters when asked to read out loud. G is a conscientious worker, neat to the point of perfection.

Socially, G is fairly well-adjusted. His friends are mostly children who are "behavior problems," but he himself avoids the tangles and scraps they get involved in. He is a good athlete. Although he obviously has friends, he sometimes prefers to stay inside (alone) and draw during break.

Drawing is a consuming pastime for G. His drawings are elaborate and detailed. He does them not for an audience or for praise, but to please himself; they are a part of his private world. G has struck his teachers as a fairly bright child who is reluctant to learn; he sometimes seems hostile; he often is so quiet and withdrawn that it is difficult to know where his thoughts are.

G's test scores reveal below-average competencies. His Otis-Lennon IQ is 88. On the Comprehensive Test of Basic Skills administered at grade 1.6, he scored an overall grade equivalent of 0.7, with reading scales in the 1-6 percentile range (0.2), language scales in the 16-33 percentile range (0.9), and math scales in the 4-25 percentile range (0.6). On the same test administered at grade 2.6, his overall grade equivalent was 1.8, with a narrower band of percentile ranges (12-26) and grade equivalents of 1.9 for reading, 1.8 for language, and 2.0 for math.

G took the KeyMath Diagnostic Profile in September of second grade and scored a grade equivalent of 1.6. He was unable to fill in blanks in counting patterns (___,6,7,___,9). He could do simple one-digit addition and subtraction problems given orally, symbolically, or in verbal problems with pictures, but he was unable to do any problems involving two-digit numbers. Although he could not tell time, his command of other measurement topics was generally above grade level. His PMDC fall test results confirmed many of these abilities. While G could count from 6 to 15, he could not count from 35 to 46. He could count backwards from 6 to 1, but not from 44. With prompting, he could count by tens from 10 to 90 and from 26 to 96. He was unable to display or interpret a display of straws or poker chips representing two-digit numbers like 60 or 37. On the other hand, using beans

he successfully solved problems like 18 + 5 and 23 - 7. He identified 4 as less than 7, 12 as more than 8, but he failed to order 2, 3, 5, and 9 from smallest to largest.

In the spring of second grade, G took both tests again. His KeyMath grade equivalent was 24. On this test he showed his greatest gains in counting skills, symbolic recognition, addition, subtraction, money and time skills. He solved two-digit addition and subtraction problems with regrouping. His success with verbal problems accompanied by pictures remained high. There was little improvement in his ability to do mental arithmetic. On the PMDC test G counted by ones forwards and backwards and counted by tens from 10 to 130 and from 26 to 106. He succeeded on all the place value tasks, using both poker chips and straws in representations of two-digit numbers. He was still able to use beans to solve addition and subtraction problems, and he answered all questions on ordering numbers correctly.

Thus on the PMDC spring test, G showed mastery of all the topics in which he had received instruction during the year. At the same time, he could not answer questions on topics outside the areas of instruction (e.g., missing addend problems), except in the specific topics of coin money and telling time.

During the year G was assigned to the group (U2) using Dienes blocks. While at first he was happy to use the blocks in his work, he came after a couple of months to regard them as unnecessary and a bother. He became quite proud of his ability to do problems "in his head" (without blocks), although he could occasionally be found using the blocks, hidden behind his work folder. Several incidents during the year suggested that he used mental images of block manipulations in his work. G worked carefully and accurately. He would struggle with a hard problem until he had it figured out, rarely asking for help, but accepting help easily. He spent much of his free time, and some that was not free, arranging his Dienes blocks in a cigar box - packing them tightly in layers, building configurations of horizontal and slanting planes.

PLACE VALUE

When G first worked with Dienes blocks in representing numbers like 14, he was very methodical in trading a long for 10 units. He would carefully place ten units in a line next to the long, so that they looked just like the long, and then remove the long. Very soon he used longs and units to represent all numbers, even when asked to use only units. In counting displays and pictures, G counted by tens and ones ("10,20,21,22"); he stumbled at first over the order of 11,12, and 13.

While G could correctly describe how many tens and ones were in a number presented with blocks, pictures, or symbolically, he occasionally reversed digits in writing numerals (96 for 69) or expanded numerals (14 _4_ tens and _1_ one). This happened especially when tens, 10-90, were introduced; writing the zero was an afterthought, often placed on the wrong side of the tens digit.

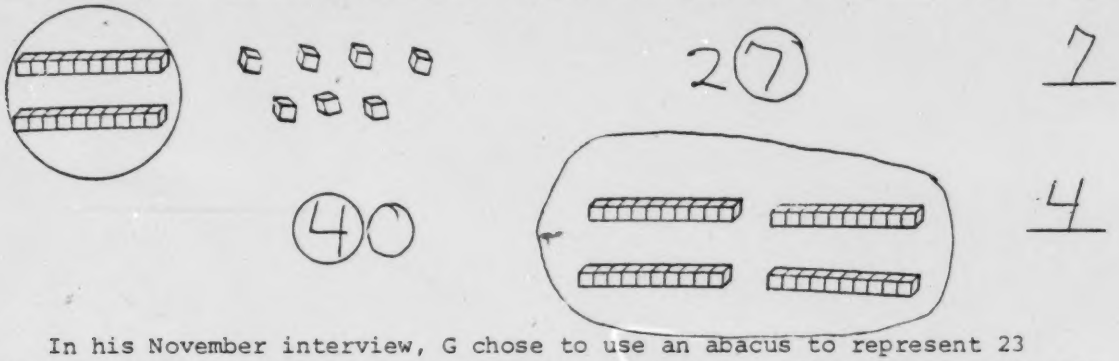
G was slow in mastering the vocabulary "tens, ones, longs, units." He was reluctant to read words out loud and recognized them by their first letters. Only after several weeks did he become fluent. On the other hand, he had almost no trouble with exercises of the form:

2 longs and 9 ones
2 tens and 9 ones
20 and 9 ones
29

when accompanied by blocks or pictures.

Later in the fall, when asked to name numbers like 67 in "another way", G showed a preference for the form "6 tens plus 7 ones". This was in contrast to the two slower students in the group, who preferred "60 + 7" (probably because it sounded more like "67", and to the two top students, who showed no preference. G was very proud of skills and insights he had gained.

One place value worksheet gave G particular trouble. He was given the numeral and a picture representation for a number like 27. In some problems, either the longs or the units in the picture were circled, and G was asked to circle the corresponding digit and write the value of the circled blocks. In other problems, one of the digits was circled, and G was asked to circle the corresponding blocks and write their value. He erred on about half the exercises:



In his November interview, G chose to use an abacus to represent 23 (flipping 20 beads on one rod and 3 on the other) and unifix cubes in rods of ten and singly to represent 32 (he chose 3 rods and 2 single cubes, without verifying that the rods actually had ten cubes).

In May, G studied three-digit numbers. He quickly made associations between the forms "17 tens" and "170". Initially, he had trouble with 20 tens, saying or writing 120 rather than 200. This cleared up when he worked with pictures of 100, 110, . . ., 190, 200 in order. When the form "1 hundred, 7 tens" was introduced, he had no trouble associating it with 170; however, when it was accompanied by a picture of a flat and longs he was no longer able to identify how many tens in all, filling in worksheet blanks with "170 tens".

As his experience expanded to numbers of less than 1000, G's skills became quite sharp. In counting by ones, tens, or hundreds with blocks, G could bridge hundreds ("386, 396, 406, etc.") without having to see or do the trade involved. Sometimes he had the longs carefully aligned so that they "became" a flat with-

out trading. Soon he was able to write a counting sequence several steps ahead of a block or picture display.

In counting backwards by tens he used his finger to cover up a row or two of blocks in a flat rather than trade the flat for longs. At no time, did he bridge into thousands correctly.

In his final interview, G demonstrated an incomplete command of three-digit place value. Shown this place value chart

hundreds	tens	ones
2	13	4

the following occurred:

G (reading): Two hundred ... two hundred thirty-four

(G is asked to show with blocks, and he displays 2 flats, 3 longs, and 4 ones.
When his attention is called to the 13, he changes the display to 13 longs)
Q: How many is this?
G: 2 hundred, 13 tens, 4 ones.
Q: Can you change it to make more hundreds?
(G trades 10 longs for a flat)
Q: How many does it show?
G: 334
(Interviewer trades the flat for 10 longs again)
Q: How many do I have now?
G: 2 hundred, 13 tens, 4 ones.
Q: Altogether?
G: 234

ORDERING

During his work with place value, G had little difficulty sorting as many as eleven pictures to show counting (40 to 50 or 100 to 200 by tens). In November, prior to receiving specific instruction regarding the concepts of more and less (but after considerable place value work), he was able to show ten more than 28 beans by adding a cup of ten beans to the display and identify it as 38. Without manipulatives, however, he was unable to say what number is ten more than 20 and said that 34 is one more than 32.

G began work with order relations by making or observing block displays for two-digit numbers and then making oral statement of comparison. At first, his statements were all of the form "41 is greater (or more) than 27." Soon he used "less" with equal facility. By the third day he stopped using blocks altogether, working either from pictures or numerals. He made few errors, even identifying oral statements as true or false accurately. When asked to correct a false statement, he retained the position of the numbers and changed the relation appropriately.

The symbols "<" and ">" were introduced with a story about an alligator who likes to eat "big numbers." G used the symbols accurately, drawing a set of teeth for several days (23<43). Often he wrote sentences (correctly) using only the greater than symbol, regardless of the left/right orientation of the display or pictures.

In a February interview (about 8 weeks after this work), he correctly identified the greater or lesser of two numbers and gave coherent reasons ("27 has only 2 tens, but this (41) has 4"). However, he made no attempt to use the symbols when asked to.

In May he showed great facility with three-digit numbers. Having only a block display or picture for a number, he was able to name the number 20 more, 20 less, 100 more, or 100 less without first changing the display. He was adept at bridging hundreds: From a picture of 87 he said 20 more was 107; from a display of 491, he said 10 more was 501.

All of his work with more and less in May was done with either a block display or picture for the first number. In the final interview the following oral dialogue took place without cues:

Q: What number is ten more than 137?
G: 187
Q: Why is 187 ten more than 137?
No response
Q: What is ten more than 37?
G: 317
Q: I want to know what number is ten more than 137.
G: I do not know.

It is evident that G depended on cues from the symbols (numerals), pictures, or manipulatives to do such problems, for without them he was lost.

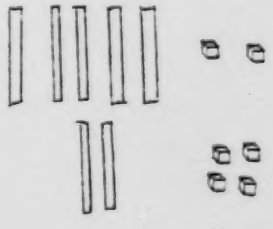
ADDITION AND SUBTRACTION WITHOUT REGROUPING

As did the other children in the group, G had trouble placing one-digit addends correctly in vertical form:

$$\begin{array}{r} 10 \\ +\ 7 \\ \hline 17 \end{array}$$

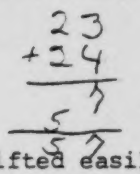
He improved rapidly, however, and made few errors in sums whether working from blocks, pictures, or symbolically.

When the form board was introduced for adding two two-digit numbers, G displayed addends accurately and gave the total after combining units and longs, usually from a visual count of the manipulatives. Gradually he shifted to a mental sum, but when pictures were introduced, he returned to counting. This was apparent in the type of errors he made:



$$\begin{array}{r} 67 \\ +24 \\ \hline 6 \\ 10 \\ \hline 76 \end{array}$$

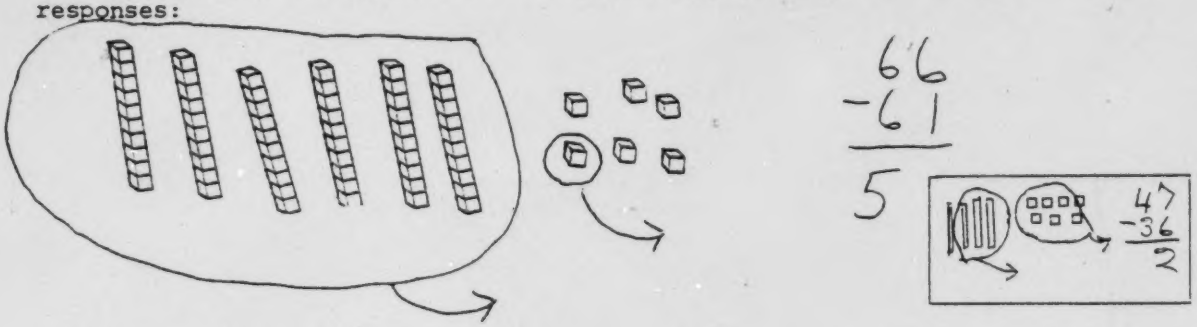
Again he gradually shifted to mental computation. He sometimes added the tens first and often omitted the zero in the tens partial sum:



As might be expected, he shifted easily to the short form for addition. He became very proud of his ability to work problems without the form board or blocks and made very few errors. He explained problems clearly; for 42 + 36 he said, "40 + 30 is 70, 2 + 6 is 8. . . 42 and 36 is 78."

In a discussion of ways to read subtraction problems ("4 take away 3," "4 minus 3," "4 subtract 3"). G became very excited when he found another way: "3 minus 4 . . . Oh . . .It doesn't work." He easily used basic facts (6-4) to solve related problems mentally (60-40) and, even when working at a completely symbolic level, did not confuse addition with subtraction.

When working subtraction problems with blocks, G made no errors, usually counting the remainder. At first with pictures he made some unusual responses:



These seemed to stem from miscounting or misinterpreting the pictures. As with addition he gradually shifted from counting the difference to subtracting mentally, and when he reached totally symbolic problems, he had no need (or desire) to use blocks.

In an interview in February G worked rapidly and accurately, explaining simply that he "added up the tens" or "took away the units." He illustrated problems with blocks accurately, and he recognized and solved orally presented addition and subtraction stories. He was able to find and correct errors in misworked problems. When he was asked to use cups of ten beans and single to illustrate addition and subtraction problems, he used only the single beans, assigning them values of ten or one according to position.

ADDITION AND SUBTRACTION WITH REGROUPING

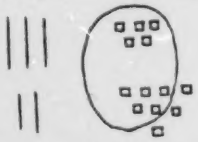
G used a variety of strategies in learning basic facts, sums 11 to 18. If the second addend was 7, 8, or 9, he mentally added ten and subtracted 3, 2, or 1, respectively (5 + 9 . . .15 - 1 . . .14). For addends of 4 or less he added on, saying the numbers to himself (7 + 4 . . .(7), 8, 9, 10, 11). Addends of 5 or 6 gave him considerable trouble; he became lost when he tried

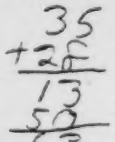
to add on, and he did not know what to subtract in using the first method. He was reluctant to use his fingers in adding on (to keep track of how much he had added), and eventually he resolved the problem when he realized that he could commute the addends (8 + 6 . . .6 + 8 . . .16 - 2 . . . 14).

At the February interview, G was asked to solve 27 +35. He added the tens mentally and wrote 5. He counted out 7 and 5 units, then the total, and wrote 2 in the ones column (total: 52). The interviewer asked him to show the whole problem with blocks, which he did. He combined the longs, then the units, and counted "50, 51, . . .59, (pause) . . ., 60, 61, 62." On a subtraction problem requiring trading (53 - 24) he wrote 30 and could not resolve the difficulty of "3 take away 4" with blocks.

The first formal instruction about regrouping involved trading activities. When G traded for more ones (2 longs, 2 units to 1 long, 12 units), he usually had to count to tell the total after the trade, but he counted the ten units all at once: "10, 20, 21, 22." With pictures he used the "ten more" pattern to tell immediately how many ones there would be after a trade.

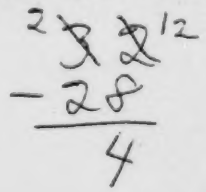
In adding two two-digit numbers requiring regrouping, G was quick to tire of the blocks. He did three problems with the blocks on the form board, following the procedure carefully, and then refused to do any more. When the problems were presented in pictures, he refused to make block displays to solve them; instead he counted the total from the picture, counting the regrouped ones as ten (45 +38: 10, 20, . . .70, 80, 81, 82, 83). At the symbolic level he used pictures when they were present, counting the ones-in-all ("10, 11, 12 13") and the tens-in-all from the picture:





As he had done earlier in the year, he gradually shifted to working the problems mentally, so that by the time the pictures were removed he had no need of either them or the blocks.

The sequence was much the same for subtraction. In working with blocks he often forgot to remove the long that he was trading for 10 units. After completing a subtraction problem on the form board with blocks, he had great difficulty recapitulating the problem: He would forget the original minuend and confuse the subtrahend with the difference. Problems presented in pictorial form were much easier for G to describe, and when blocks or pictures were coupled with the symbolic algorithm, G became very confident of the process. He soon refused to use the form board, although he used blocks surreptitiously to help himself with trades and some subtraction facts. He explained his work with few words:



G: "I traded. 12 take away 8 is 4."

G distinguished consistently between trade and no-trade subtractions. He was less careful to distinguish between addition and subtraction, especially when the only cue was the operation sign. However, he began to use a short form for addition with regrouping, doing the entire problem mentally recording only the answer:

$$\begin{array}{r} 47 \\ +29 \\ \hline 76 \end{array}$$

G: I put the 6 down here and I put the ten up here (pointing to the 4). Then 4 and 2 more and 1 more makes 7.

At his April interview, G worked problems using both blocks and beans. He exhibited competence and confidence, doing most of the calculating mentally. His explanations were clear and seemed to be based in his work with manipulatives:

$$\begin{array}{r} {}^{6}\not{2}\not{5}^{15} \\ -\ 38 \\ \hline 37 \end{array}$$

G: I didn't have enough ones to get the 8 out, so I took a ten and got ten ones. Then I crossed out and put the 15 there (pointing) and the 6 there.

At his final interview G transferred his skills with two-digit addition and subtraction to three-digit problems that required regrouping. He had not faced such problems before. Using both blocks and beans he solved several problems successfully. He also worked a subtraction problem without aids:

$$\begin{array}{r} {}^{1}\not{2}\ \not{3}^{13}\ 7 \\ 1\ 7\ 6 \\ \hline 61 \end{array}$$

G: I had to trade. I traded that for 10 tens . . . I traded one hundred for 10 tens.

Of all the children in the Dienes block group, G seemed to benefit most from the progression from blocks to pictures to symbolic work. He used the blocks carefully when first learning a new concept, but also very proud of not needing them once he understood the process.

VI THE CASE OF H

by

Stewart Wood

BIOGRAPHICAL AND TESTING INFORMATION

H was born December 8, 1968 and is an only child. His parents are separated. H lives with his mother, who is completing her undergraduate degree at a nearby university.

H attended kindergarten and was in the Title I program in first grade. As a December baby he is one of the youngest children in his class. He began reading at the end of first grade and throughout second grade has used free time in class to read.

H likes to do things, actively, and not be judged right or wrong on the work he does. He constantly talks to himself, working out loud. If interrupted and told he is doing something incorrectly, he reacts: "All right! I know, I know!" erases what he has done and starts all over, usually incorrectly. In class work he often tunes out oral directions, then begins working by doing what he thinks is right. When reading, he skips words he does not know or makes up words in their place. He thinks of himself as both independent and competent, finding it difficult to accept guidance from a teacher or from peers. On most tasks he works rapidly and is among the first to finish.

He is a loner--a sociogram of his second grade class showed no mutual friendships: one child named H as a friend, and H named only one (different) child as his friend. He is TV oriented. He was absorbed by Electric Company during the Title I program, and his conversation is noticeably more filled with references to TV shows than that of his peers.

H's test scores reveal below-average competencies. His Otis-Lennon IQ is 86. On the Comprehensive Test of Basic Skills administered at grade 1.6, he scored an overall grade equivalent of 0.9, with reading scales in the 25-45 percentile range (1.6), language scales in the 8-25 percentile range (0.2), and math scales in the 1-5 percentile range (0.1). When the test was administered at grade 2.6, H showed well over a year's growth in his weakest areas. He scored an overall grade equivalent of 2.4, with reading scales in the 36-41 percentile range (2.4), language scales in the 31-75 percentile range (2.6), and math scales in the 14-17 percentile range (1.8). Overall, although his IQ is low, H shows a capacity to learn and a potential for average at-grade-level performance.

KeyMath, administered at grade 2.0, yielded a grade equivalent of 0.5 and showed some familiarity with concrete and oral addition and subtraction, coin money, and an inch ruler. H was unable to complete any written problems (like 1 + 3). On the PMDC fall second grade test, H was able to count animals in a set and to count back from 6 to 1 without prompting. He was able to count from 6 to 15 by ones and from 10 to 100 by tens with

prompting. He showed no understanding of two-digit place value. He was able to demonstrate and solve 2 + 4 = □ using beans, but not 7 - 4 = □ .

In May of second grade, H was given each of these tests again. His Key Math grade equivalent at this time was 1.5. H showed some improvement in his recognition of symbols (+, -, =), his counting skills, and his familiarity with coins and telling time. His lack of success with oral problems (3 + 2 = □) and "story" problems was little changed. He was able to do written addition and subtraction problems only if they did not involve regrouping. On the PMDC test H was able to count by ones forward and backward with only occasional prompting. With prompting he could count by tens from 10 to 100 and from 26 to 96. In a series of questions using straws and poker chips to represent two-digit numbers, H failed to display or interpret correctly displays for 60, 50, 37, 45, and similar numbers. Using beans, H was again able to demonstrate and solve 2 + 4 = □ or harder problems. At the time of this spring testing, H was able to identify which of two numbers was more, but not which of two numbers was less. Particularly on the place value, addition, and subtraction tasks in these two test, H's performance fell short of his work earlier in the year, both in individual interviews and in classwork.

As a second grade student in the T1 teaching experiment, H was assigned to the group using Dienes blocks. Among the six children in this group, he was most verbal and the most restless. He was usually the last to master a new concept and often could not handle at a symbolic level concepts or procedures he had apparently mastered at a manipulative-verbal level. At the same time, there were many instances of his verbal-manipulative work consisting of rote mimicking of teacher-demonstrated procedures and statements.

PLACE VALUE

One of the first characteristics H showed in working with Dienes blocks was his apparent command of isolated facts coupled with his inability to use these facts in answering questions about blocks. For example, even though he could make the verbal statement that "10 and 10 is 20" and could identify a long as having 10 units, he resorted to counting the unit marks by ones to find how many units there are in two longs. Sometimes confusion seemed to lay in the words "ten" and "long"; "long" was the name of a stick that had ten units but "ten had no concrete referent:

(with 2 longs and 5 units displayed)
Question: How many tens are shown?
H: 20
Q: How many longs are there?
H: 2

In determining how many blocks together in a mixed display of pictures of longs and units, H went through three stages of response. At first, he counted the unit marks of the longs plus the additional units all by ones. This method led to errors due to the inaccuracy of his pointing and mistakes

in his counting. After about a week he began to count the longs by tens, but could not shift from tens to ones appropriately (for example, counting 3 longs and 2 units as "10, 20, 30, 40, 50"). This stage lasted several weeks and included work with numbers 11 to 20, tens 10 to 90, and random numbers to 99. When asked to display a number during this period, he would often not distinguish between longs and units (for 38 he showed 11 longs). Toward the end of this stage, as he began to distinguish more consistently between longs and units, he still made counting errors (for a picture of 52 he counted "10, 20, 30, 40, 50, 21, 22"). In the final stage, H counted longs and units separately, then combined his results (for 82, he counted 10, 20, 30, . . ., 80, 1, 2, 82). H developed this approach after extensive work with place value charts, counting displays and ordering pictures in specific decades (e.g., 80 - 90), and worksheet exercises such as:

4 tens + 2 ones

40 + 2

42

Even in this stage, however, H continued to confuse tens and ones; for example, when asked to show 87 with blocks he made a correct display, but said that there were 7 tens and 8 ones in his display.

In written work, H was largely dependent on rote patterns. If blanks were arranged in an ordered pattern he had little trouble, but with random arrangement he wrote:

5 tens	50
10 tens	70
90	10 tens
60	6 tens
60	60 tens

This worksheet had no pictures, so H had no cues other than pattern and the word "tens." In writing expanded numerals H was able to use some patterns but not others:

"3 tens + 5 ones" is written on the board, H makes a correct block display and writes:

3 tens + 5 ones

3 + 9

35

"80 + 3 is written on the board; he write:

80 tens + 3 ones

80 + 3

803

H had more difficulty in tasks involving partitioning a number into tens and ones than vice versa. He could say that 1 ten and 3 ones makes "13 altogether," but could not say that 13 is made up of "1 ten and 3 ones."

Throughout his work there were some things which H did with little difficulty. When asked to write 2-digit numerals in a place value chart (oral question, no block display), he could do so without reversing digits. Thus from an oral stimulus he distinguished between, say, 14 and 41. When asked which blocks showed "the tens" or which blocks showed "50", he responded correctly. In general, he performed well in tasks where he was not dependent on his own counting skills and had only a simple matching or choice to make.

In the November interview at the end of this work with place value, H showed no ability to respond to questions not phrased exactly as the instructional tasks had been and no ability to offer explanations for the responses he could make. He could not say that the 5 in 53 means "5 tens." When given cups containing ten pieces of candy plus some individual pieces, he represented 27 with a pile of 2 pieces and a pile of 7 pieces. (When asked to show "one more" he made piles of 2 and 8 pieces and said there were 82 in all.)

H would seem that his first work with blocks, in which he persisted in counting the units in longs (often inaccurately) indicated his non-readiness for place value work. He learned later that 5 longs was 50 from an oral-rote pattern, not apparently from a belief that 5 longs contained 50 units "glued together for convenience." Observing H at work without manipulatives could be quite misleading: He could write 2-digit numbers from dictation; he could answer specifically phrased questions ("How many tens in 53?" but not "what does the 5 in 53 mean?").

One wonders exactly where H stood with respect to number conservation. In counting blocks he often recounted each time in answering a series of questions, even though in theory he had the information available from previous counts. Yet at other times he moved and mixed blocks easily, without disturbing his sense of how many were displayed. Did his early preference for counting individual unit marks in longs stem from a mistrust of how many units there were (understandable, given his inaccuracy) or simply from his associating the question "How many altogether?" with counting by ones, because he did not know how to count any other way?

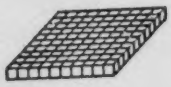
In May, H worked with 3-digit numerals. He immediately picked up (by rote) that 100 is "10 tens," but during the month he made such trades as a flat for 13 longs, 10 longs for 10 units, a flat for 4 longs. For the first

two weeks, he counted displays of longs only as "10, 20, 30 . . .," even in response to the questions "How many longs are there?" Having counted a display of 15 tens ("1, 2, 3 . . .15") he could not say there were 150 in all without counting again ("10, 20 . . ., 150"). By the second week of work he could say that 15 tens is "150 in all," but even at the final interview he could not say that 150 contains 15 tens.

In written work H again relied on rote patterns. Questions involving expanded numerals were always accompanied by block displays or pictures; still H had great difficulty:

100 tens	15 tens	130 tens
1 hundred, 0 tens	1 hundred, 5 tens	1 hundred, 3 tens
100	105	013

In the last two weeks of work, H was able to make correct block displays using flats, longs, and units for random 3-digit numbers from either an oral or a written stimulus. He could say that the 6 in 654 means "6 hundred." Yet he was cavalier about position, writing 651 for 561, 805 for 508. Zeroes were particularly troublesome:



hundreds	tens	ones
100	0	4

1004

H never developed a sense of numerousness for large numbers nor became comfortable with the sound of number names. With 6 flats in front of him he was asked how many units would be a fair trade; he replied "116 . . .no 160 . . .I mean 60." He was completely stymied by non-standard displays such as 2 flats, 13 longs, and 4 ones. He could neither count the display nor trade 10 longs for a flat to make a standard display. He could count by ones, tens, or hundreds from a number like 487 only by making a block display for the number, adding a unit, long, or flat and counting the new total--all under considerable guidance.

ORDERING

Several times during the fall H was asked to put ten or eleven pictures in order to show counting. These were more pictures than he could cope with: he chose a picture at random, counted it (often inaccurately), discarded it, and picked another. Occasionally he stumbled onto pictures of consecutive numbers and recognized them. He never developed a strategy of just counting the units in pictures for a decade, as did all the other children in the group. After much assistance, when he had pictures of consecutive tens in order, the pictures helped him to learn to count by tens and to make associations between the forms "7 tens" and "70."

H had no apparent skills with the concepts of more and less as he began detailed study of the topic in December. At first, when asked which of two displays of blocks had more, he counted the blocks in each display. Only if he happened to count the smaller display first did he recognize the other group as more. If he counted the larger group first, then after counting both groups he had no idea which was larger. This behavior seemed to stem from poor number memory: Having already verbalized the smaller number (say, 37), H recalled it as he counted past 30 in the larger display; but when the order was reversed there was nothing to prompt his memory of the first (larger) number while he counted the second.

Gradually H began to focus on which display or picture had more tens, although he continued to count each display. While he became adept at identifying which display had "more," he could not make a complete statement comparing the two numbers. Often he said "23 in more is 15" or some variant. The words did not make sense to H; he did not perceive "is more than" in the phrase "23 is more than 15" as giving information about the numerousness of 23, compared to 15.

When H had to choose between using the phrases "is greater than" or "is less than," he guessed. When the symbols were introduced, he often drew triangles ($\triangle$). In tasks that began with either pictures or numerals, H usually displayed each number with blocks; even with a pair like 60 and 62 he had to "make" the numbers with blocks before he could say, haltingly, "60 is less than 62." Without this action, his responses seemed to be random guesses.

In May, H was asked to display a number, then show what 1 more (less), 10 more (less), 20 more (less), etc. would be, and tell the new number. H confused "more" with "less," 2 with 20, etc. For the first two weeks he counted the new display from zero, as though he had forgotten how many there were originally. Gradually he had to recount only the blocks he had altered (the longs for "10 more than 341"). After considerable practice in counting by 1's, 10's, and 100's with blocks he was able to show 20 more than 71 and get the new total by counting on ("81, 91"). If the new number involved bridging hundreds (20 more than 286), H neither recognized the need for a trade nor was able to perform the trade without guidance. When he tried to count without trading, he miscounted: "276, 286, 296, 226."

In his final interview H was asked what number is ten more than 137. He answered "138. . .because it is next." For the number that is ten less than 253, H said "254."

VII. THE CASE OF J

by

Judy Voran

BIOGRAPHICAL AND TESTING INFORMATION

J is a male child, born October 11, 1968. He lives with his mother and father and baby sister in the married student housing of a nearby university. His father is a foreign student in graduate school, and his mother is a housewife. There seems to be a close relationship among family members and the birth of his sister in the late winter brought much excitement for J. He had informed the group months before that he had a baby sister, then when asked what her name was, he replied, "I don't know, she hasn't been born yet!"

A language barrier seems to be J's only school problem, and much of this has already been overcome. J is a native of Iran, and Arabic is spoken in the home, since his mother speaks very little English. J attended another public school in the United States for half of his first grade year and began second grade in the school at which the experiment was conducted. At the beginning of the year he talked very little in the classroom, but soon became relatively fluent verbally. Other children never made fun of J's accent, perhaps because there are quite a few children at this school with foreign accents as well as many with Black dialects. He has very good reasoning ability but some times encounters difficulty with his verbal explanations. Some cultural problem arose early in the year with regard to holidays, but J adjusted very well to his new situation in America.

J has a strong self-concept and is confident of his abilities. He is an independent child, a conscientious worker, and has been given many responsiblities at home. His father worked with him nightly, mainly in math and Arabic, to keep him familiar with the language of his country to which the family plans to return. In the classroom J was a popular boy who rarely got in arguments but defended himself when necessary--with words rather than fists. He always gave a reason for everything he did, and it was usually a very intricate one.

On the Comprehensive Test of Basic Skills (CTBS) administered at grade 2.6, J scored 2.3 grade equivalent in reading, 2.6 grade equivalent score in language and 3.2 grade equivalent score in math. He was at the 77th percentile nationally and the 83rd percentile locally. On the math computation subtest he scored at the 68th percentile, and on the math concepts and application subtest he scored at the 86th percentile, based on national percentiles. On the Otis-Lennon test given at the beginning of his second grade year, J obtained an IQ of 90. This group, standardized test IQ was much depressed due to J's language problems, and is not nearly a true picture of his mental capabilities. Because of his language problems, J participated in the county's Title I reading program and received

instructional emphasis on vocabulary and usage. He will not participate in the program next year, since his reading scores are now above the maximum allowed by the guidelines.

On the PMDC 2nd grade test administered in the fall, J was able to count sets fairly well, miscounting only one set. When asked to count backward from 44 to 25, he got to 30 and said "20." He counted by tens from 10 to 100, then began counting by hundreds (80, 90,100, 200, . . ., 800) and exhibited the same behavior beginning with 26, counting by tens to 106, then 206, 306. When shown bundles of straws and single straws to represent tens and ones, J counted the bundles as single straws and said "10" instead of 37. However, when shown colored chips which represented tens and ones, J counted the display by tens and ones correctly and said "43". When asked to construct a set to represent a given number, J used both straws and chips correctly. On the addition and subtraction problems J answered only one of four items correctly, using counters for some items only. He correctly answered all of the ordering items, telling which of two numbers was more or less.

On the spring administration of the PMDC test, J counted backward from 44 to 41 and could go no further. He was successful on all other tasks, except for two items on class inclusion.

J had a grade equivalent of 1.8 on the KeyMath test given in the fall. Relative strengths were in geometry and symbols, money, and division. Relatively weak areas were subtraction, mental computation, and missing elements. J solved 3 written problems, all without regrouping.

On the spring administration of the KeyMath test, J's grade equivalent was 2.8. Relatively strong areas were numeration, subtraction and numerical reasoning (example: 5 +△= 9; 7 - △ = □). Weak areas were missing elements, fractions and measurement. J was able to solve all one-digit and two-digit addition and subtraction problems, with and without regrouping. He also answered 2 written multiplication fact problems.

As a second grader in the T1 teaching experiement, J was assigned to the single embodiment group using the abacus (U3). There were three boys and three girls in the group and J worked well with all the children, though one boy and one girl in particular were usually his first choice for partners. J was almost always the first to finish any work, whether enactive, iconic or symbolic in nature. He showed interest in working with the abacus when it was first given to him, or when new material was first introduced, but tired of it quickly and wanted to move to a completely symbolic level as soon as possible. At times he became impatient when having to wait for slower children in the group to finish or when given another example of something he already understood. However, he seemd sympathetic to children not as capable as he was, and many times offered to help them when they were having problems. When J grasped a new concept, he seemed almost to bubble over with excitement, laughing, clapping his hands together, standing up, and talking quickly and loudly. He truly exhibited the "joy of learning" in these spontaneous moments.

The following discussion of J's performance in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

J's place value concepts were well developed. He displayed an immediate understanding of tens and ones and could show any two-digit number on the abacus less than two weeks after the experiment began. He could also read any two-digit numeral. The first day the abacus was introduced and tens and ones were used to show a number, J showed the experimental teacher his abacus with 847 displayed, though the hundreds rod had not been mentioned in class. He said, "Let the red ones be hundreds (pointing to the hundreds rod) and these are tens and these are ones (pointing to the appropriate rods). "Now, tell me what this number is." When asked if he knew what it was, J said "No, I don't know this one. It's too hard!" Months later when three-digit numerals were introduced, J felt vindicated to find out his October intuition was correct.

J consistently explained the meaning of two-digit numerals in terms of the number of tens and ones. For 53 he said, "this is tens (pointed to the 5) and this is 50 and this is ones (pointed to the 3) and this is 3." He was able to fill out forms such as:

_____ and _____ones

____tens and ____ ones

_____

using pictures, but he preferred to work only with symbols when possible. Occasionally, he filled in a form as:

5 tens and 6 ones

5 + 6

56

As soon as he was asked to read it, he was aware of his mistake and corrected it. In November, after finishing a worksheet of this type one box was left over; so he filled it in as follows:

90 tens and 60 ones

900 + 61

961

Somewhat surprisingly, J experienced some problems initially with three-digit numbers. He had some difficulty counting by tens past 80, hesitated on 90 and 100, and then began counting by ones instead. After three days this counting difficulty cleared up, although in late May when the group was counting a display of 1 hundred 9 tens and 10 ones, the trouble flared up. J accepted this display as showing 200, and so the group counted as more ones were added to it. When the group added the tenth one after 200, J said "300" and another boy said "210." He argued at great length for this choice of 300 and was not convinced until he saw the display of 200 + 10.

J could easily trade to show numbers such as 150 as 1 hundred 5 tens or as 15 tens, and he felt confident in stating 170 = 17 tens for numbers 200 or less, and for multiples of one hundred. When asked how many tens for a number such as 374, he was unsure unless he saw it displayed as 37 tens and 4 ones. When he saw it displayed as tens and ones, he knew immediately there were 37 tens and did not have to count the tens, but if he saw it as 3 hundreds, 7 tens and 4 ones, he had to go through a much longer mental process in order to say it had 37 tens.

In the final evaluation interview, J was shown

hundreds	tens	ones
2	13	4

and was asked to read it. He read it as "2 hundreds and 13 tens and 4 ones"; then he was directed to show that number on the abacus. He said, "I know that----it's 334," and then showed it on the abacus. Later in the interview he was asked to read the following and determine if it was correct:

245 = 1 hundred 13 tens 15 ones

He talked to himself some and touched the 1 in 13, moving his finger over the hundred blank, then touched the 1 in 15, moving his finger to the tens blank. Finally he looked up and said matter-of-factly, "Sure,that's right."

He was shown a picture of 14 circles representing tens and 16 squares representing ones and then was asked if the picture showed the numeral on a card (156). He counted the tens to 100, covered them up and then counted "10-20-30-40" and said,"No." He was directed to count the ones, which he did correctly, but he still said the picture did not show 156, it showed 140 and 16. The interviewer then asked what 140 + 16 is, and J laughed and said, "Oh, that's right."

ORDERING

Ordering pictures was a task easily and quickly done by J. He ordered pictures for a decade (80-90) and a mixed set (23, 43, 34, 75, 90, 7) with equal ease. Early in the year he was able to choose the greater of two numbers such as 19 and 25, and he disliked having to show the numbers on the abacus to verify what he already knew to be true. He and the other boys in

the group came up with the generalization for determining which number was greater: "Look at the tens and the one with more tens is greater; if tens are the same, look at the ones and the one with the most ones is greater." By early December he was consistently giving the number of tens as a rationale for his choice of the greater or lesser number.

Use of the > and < signs caused no difficulty for J, and it seemed as if he had worked with them before. He enjoyed making number statements with them to be judged as true or false. In December he showed his T1 teacher what he had written, 8106457 < 9017568, then asked if it was true. When the teacher asked him the question he said "Yes, because 8 is less than 9 and 1 is with 0 and 0 is with 1 and 6 is with 7 and 4 is with 5 and the 5 is with 6 and 7 is with 8," pointing to each pair of corresponding digits as he mentioned them. He may have thought it was necessary for each digit in the left numeral to be less than its corresponding digit in the right numeral. J also made up a card with 900 < 10,000 and said it was true because the number on the right had more zeroes.

When asked in an evaluation interview in November to show 27 candies using cups with 10 candies each and single candies, he did so. However, when asked to show more, he moved over another cup and said "37." Then when asked to show 10 more he was puzzled, pointed to a cup and said,"10 more of these?" He could not see how to show it and finally said, "It would be 37." When asked to show it with the candies, he moved over all the cups on the table and said, "That's 57."

During the same interview J was asked to write the number that is 10 more than 20 and he asked "Write twenty?" then "Write 3?" and finally wrote 3. When asked to write the number that is one more than 32, he wrote 1, and when asked to write the number that is 1 ten more than 30, he wrote 4. He was then asked to read each numeral, and the questions were repeated with J responding orally. He correctly responded to 10 more than 20 and 1 ten more than 30.

When three-digit numbers were introduced, J was able to apply his concepts of more and less to them. He ordered pictures of three-digit numbers and wrote the number which was 10 more, 2 less, etc., with the aid of the abacus. He seemed to enjoy using the abacus more than he had in months, especially to show a number that was more or less which required trading on the abacus. When first filling in blanks to show counting by ones, tens and hundreds, both forward and backward, J used the abacus only when regrouping and when counting backward. Two days later when filling in a similar worksheet, he abandoned the abacus completely and correctly wrote the counting sequences.

On the final evaluation interview in June, J was asked to tell the number which is 10 more than 137. He replied "147, because you add one more to three tens and it makes four tens." When asked to tell the number that is 10 less than 253 he said "243," and explained it by saying "You take away one from the tens."

ADDITION AND SUBTRACTION WITHOUT REGROUPING

J was able to work addition and subtraction problems on the abacus but preferred to work with the symbols only. He had no trouble writing problems from a picture, but always reversed any problem for a picture which showed a 1-digit addend over a two-digit addend, so that instead of

$$\begin{array}{r} 5 \\ +\ 20 \\ \hline \end{array} \quad \text{he wrote} \quad \begin{array}{r} 20 \\ +\ 5 \\ \hline \end{array}$$

In December, an effort was made to relate addition facts, such as 5 + 3 = 8, to larger numbers, such as 50 + 30 = 80. J was very proficient with the basic facts, and he began to think in terms of <u>larger</u> numbers, though he got a little mixed up on the fact involved. He said "What is 300 plus 800?" then he looked at his fingers for a moment and said "That's eleven-hundred!" Two days later he looked up at the chalkboard and saw

$$\begin{array}{r} 5 \\ +\ 3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 50 \\ +\ 30 \\ \hline \end{array}$$

written on it. Suddenly he said "Oh, see, see! That is 5 and 3 and that is 5 and 3, too. I see, I see. That the same and the answer is the same." Written symbols seemed to have a much greater impression on J than even the enactive and iconic work he did. Two more days after this "discovery," J filled up the back of his worksheet with the following problems:

$$\begin{array}{r} 1000 \\ +\ 2000 \\ \hline 3000 \end{array} \qquad \begin{array}{r} 500 \\ 6000 \\ \hline 11,000 \end{array} \qquad \begin{array}{r} 7000 \\ +\ 9000 \\ \hline 16,000 \end{array} \qquad \begin{array}{r} 9000 \\ 9000 \\ \hline 18000 \end{array}$$

J refused to show partial sums unless specifically told to do so. He carried out the following procedure when the group was learning to add tens and ones:

1) $\begin{array}{r} 25 \\ +\ 13 \\ \hline 8 \end{array}$ added 5 and 3 and said "8," wrote 8 in the ones column.

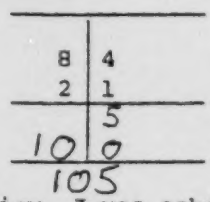
2) $\begin{array}{r} 25 \\ +\ 13 \\ \hline 38 \end{array}$ added 2 tens and 1 ten and said "30," but instead of writing 30 on the next line, he wrote 3 in the tens column.

He also balked at adding ones before tens, saying that it slowed him down because he came to the tens first.

J was the first student in the group to come to a problem on a worksheet which had been included by mistake. He had worked the problem like this:

	8 | 4
+	2 | 1
	5
	1 | 00

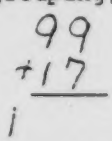
but he knew this was not correct, so he asked his teacher for help. He told her the answer was 105, but did not know how to write it. His teacher asked him how many tens was 8 tens + 2 tens and he answered 10, so she told him to write 10 on the tens side. With that help, he changed the problem to:



During an evaluation interview, J was asked to solve 8 addition and subtraction problems without regrouping. He read each problem and solved each correctly. When asked to explain how he got his answer he said, "I added the ones first and then the tens," pointing to the appropriate digits. He was asked to solve mentally problems such as 24 + 32 and 45 - 32, which he did correctly except for one counting error.

ADDITION AND SUBTRACTION WITH REGROUPING

J quickly adjusted to trading on the abacus to show more ones and less ones, although at times he would forget to bring over a ten after having turned over 10 ones. His preference was to work only with symbols, though he could and would show the steps on the abacus if asked to do so. Three days after beginning work with addition with regrouping, J wrote:



He had added tens and said "It's one hundred," then had written a 1 in the hundreds place, but did not know how to finish it.

After 8 days of working with addition problems with regrouping, J tired of showing partial sums and he answered all problems in one step, writing the tens digit first. When completing worksheets with pictures, if the problem was already written, he totally ignored the picture. If he was asked to write the problem, he used the pictures to write the addends or the minuend and subtrahend, but then he computed without using the pictures and dealing only with his written symbols.

J had some difficulty with problems such as:

$$\begin{array}{r} 90 \\ -\ 37 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{r} 20 \\ -\ 8 \\ \hline \end{array}$$

He answered them as if 0 - 7 = 7 and 0 - 8 = 8, not seeming to realize that regrouping was necessary. On an evaluation interview in April, J was shown

a card with:

$$\begin{array}{r} 43 \\ -\ 26 \\ \hline 23 \end{array}$$

and was asked if it was correct. He answered yes, because you could change the 6 and the 3 around. However, J never made this type of error when working subtraction problems. During that interview J was given 2-digit addition and subtraction problems with regrouping to solve, some with an abacus or beans and some with symbols only. His usual method was to show the addends or the minuends on the abacus, or with beans then compute the answer looking at the written problem, and then finally adjust the manipulative display to correspond with the answer he had mentally computed. When asked to explain his method on written problems he had solved, he explained it in terms of crossing out numerals and making the ones 10 more and the tens 1 less, although he had actually written nothing down except the answer.

On the final evaluation interview, J was given 3-digit addition and subtraction problems with regrouping to solve. He correctly solved 6 of the 8 problems, and 1 more when he was instructed to trade to find the answer. Of those he correctly answered, he worked 3 of them at the symbolic level first, then showed the answer on the abacus or with beans.

For the other problems which were "too hard," he went to the manipulative aid first and worked the problem. For

$$\begin{array}{r} 245 \\ -\ 148 \\ \hline \end{array}$$

J showed 2 hundreds, 4 tens and 5 ones. He took away 1 hundred and 4 tens. Then he traded the remaining hundred for 10 tens, and one of those tens for 10 ones. He then removed 8 ones and said the answer was 97.

VIII. THE CASE OF K

BY

JUDY VORAN

BIOGRAPHICAL AND TESTING INFORMATION

K is a male child, born November 28, 1968. He lives with his parents and brother who is three years younger. K rides a school bus and lives in a mobile home park. His father is a security officer at a nearby university and his mother is a housewife who is very involved in the school as a room mother and volunteer. K has a good home life and he talked excitedly about his family's vacation in the spring and of the many trips he and his father took. He knew his parents were deeply concerned about him and one day when he became ill at school, he asked to call his mother saying, "I know she'll be worried about me and will want to come and get me." She did.

All of K's school years have been at the same elementary school. He attended kindergarten and participated in the county's Title I reading program and ESAA math and reading program. In the classroom he read in the middle reading group. K is left handed and experiences difficulty when learning to write. His handwriting is poor and he frequently reverses letters and digits. At times this causes him frustration. One day he tried to write 3. He first wrote S, then wrote E over it, finally wrote C on top of that and said, "I keep on making S's and E's!"

K is a very popular boy in his class and was one of the 8 children most frequently chosen on the class sociogram. He shows concern for others and is sympathetic to children who are having trouble with their work. He feels that it is very important to be the first to finish an assignment and may times makes careless errors on written work in order to do so.

On the Comprehensive Test of Basic Skills (CTBS) administered at grade 2.6, K scored a grade equivalent of 2.2 in reading, 2.4 in language and 2.4 in math. Percentile ranks on the math computation subtest and math concepts and application subtest were nearly the same, with a total math ranking of 41st percentile nationally and 53rd percentile locally. He obtained an IQ of 90 as measured on the Otis-Lennon administered at the beginning of his second grade year.

On the PMDC second grade test administered in the fall, K was able to count sets by touching pictured objects. He could count from 6 to 15 and from 35 to 46, but at times counted out of sequence (38, 40, 39, 40). He counted backward from 6, but not from 44, and he counted by tens from 10 to 80, skipping 20. When telling how many in a display of tens and ones when straws were used, he counted each individual straw in each bundle, but made counting errors, so the answer was wrong. When shown a chip display, he counted each chip as one, not differentiating tens and ones. When asked to construct a set, he either made no attempt or said there were not enough chips. K used counters to solve an addition fact problem used counters incorrectly on a subtraction fact problem and made no attempt to solve problems with 2-digit numbers. K correctly chose which was less and which was more from (7,4) and (8,12) but not from (19, 31). He incorrectly ordered four 1-digit numbers from smallest to largest.

On the spring administration of the PMDC test, K showed much improvement. He was successful on all counting tasks except for counting backward from 44, counting by tens past 100, and counting by tens starting at 26. He could tell the number of a display of tens and ones, and could construct displays to show a given number. He used counters correctly on all addition and subtraction problems, and answered all order items correctly.

On the fall KeyMath test, K scored a grade equivalent of 1.6. Numeration and subtraction skills were relatively weak, and he could answer no questions pertaining to missing elements. Relative strengths were the areas of division and measurement, and K correctly answered two written addition problems.

K scored 2.3 on the KeyMath test in the spring. Word problems and mental computations were relatively strong areas, and missing elements and measurements were relatively weak areas. K correctly answered 4 written addition problems, one of which involved regrouping, and 2 written subtraction problems, neither of which involved regrouping.

As a second grader in the T1 teaching experiment, K was assigned to the single embodiment group using the abacus (U3). There were three boys and three girls in the group, and K was very attached to one of the boys. K attempted to compete with this student, who was much quicker and more capable than K. This situation created problems because K would rush through his work, writing sloppily and at times illegibly, so he could finish when his friend did.

K was emotionally immature and became frustrated when he could not do something correctly. He did not want to wait for instructions and felt that he already knew how to do many things which he could not do without direction. At times K became so frustrated, either because he did not understand something or because he could not write something, that he cried. After being shown how or after being given a clean paper to write on, he would start over and usually successfully complete the task.

The following discussion of K's performance in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

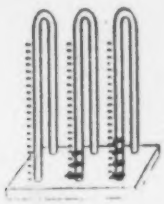
PLACE VALUE

Reversal problems interferred with K's developing place value concepts. The following are some problems caused by his perceptual difficulties:

23 = 02 and 3 ones , 34= 13 and 4 ones

18 = 8 tens 19 = 9 tens 41 = 4 tens

He also showed 13 on the abacus for 30 and 25 for 52 occasionally, but errors of this type were not as common in the enactive stage. The following illustrates a combination of frequent errors made by K at the symbolic level:



13 and 4 ones

3 tens and 4 ones

43

K was able to read most two-digit numerals and only experienced difficulty with the numbers 12-19. When asked to explain why 53 was "53" he could only do so by saying that a 5 in front of the 3 made it 53, and if there wasn't a 3 it would be just 5. Counting an abacus or picture

display presented initial difficulty, however, K would begin counting the tens beads by tens but could not switch to counting by ones when he moved to the ones rod. At times he would begin counting by ones while still counting tens, so that a display for 35 could end of being counted as "10-20-30-40-50-60-70-80" or as "10-20-21-22-23-24-25-26."

Numbers of greater than one hundred created more reading, writing, and counting problems. K could count by tens to 100, but then began counting by ones and took a long time before being able to count "80-90-100-110-120." K could not write a three-digit numeral without showing it on the abacus first and going through any trading that was necessary. He also could not count a display showing 10 or more tens or ones. When asked what number was represented by 2 hundreds, 13 tens and 4 ones, he said "It don't make sense, I'll have to trade." When pressed for an answer he finally said "234," then traded and said "Now it makes sense --334." He showed 2 hundreds, 4 tens and 5 ones on the abacus and said it was "245." Then he was told to trade for more tens, which he did, but when asked what number was now shown, he again commented that it didn't make sense and finally said "165."

K had a very difficult time writing numerals, and when counting by ones from 100 he wrote 100-1001-1002- . . .1010. For counting by tens he wrote 100-112-120-131-125- . . . .He could not count orally by tens from 100 after having counted by ones from 100, and vice versa. He frequently made writing errors such as:

119 for 190, 220 for 200, 116 for 160 and 105 for 150.

K became very confident of his ability to express multiples of 10 < 200 as 1 hundred and some tens or as all tens. He did not count out 10 tens for trading, but could easily change from 1 hundred, 4 tens to 14 tens. For multiples of 100, he came up with a "trick" for telling how many tens. He said it was just whatever was in front, meaning everything except the last zero, 30 for 300, 40 for 400, etc.

Numbers larger than 200 and non-multiples of 100, especially those with a non-zero one digit, were not as easy for him to describe in terms of tens. He could trade on the abacus to show 265 as 26 tens and 5 ones, and after counting the number of tens he could tell that 265 had 26 tens, but he could not see it as 2 hundreds, 6 tens and 5 ones and come up with 26 tens.

ORDERING

A major difficulty K encountered in this area was due to his writing and reversal problems. When asked during a November evaluation interview to write the number which is 10 more than 20, he wrote 50 . For the number which is 1 more than 32 he wrote 33, and for the number which is 1 ten more than 30 he wrote 13, probably thinking he had written 31 to show 1 more than 30. He had an especially difficult time with the teen numerals (12-19).

When ordering picture sets in a decade; 80 to 90 for example, he could arrange the pictures showing 81-89 in the correct order, but would put pictures for 80 and 90 together, sometimes before 81 and sometimes after 89. Ordering a mixed set of pictures (23, 43, 34, 75, 90, 7) proved to be a much more difficult task for K.

In December K became more confident of his ability to choose the greater or lesser number. He also helped the other two boys derive the "rule" for determining which number was greater; by looking at the tens. This rule proved helpful to him and he was able to choose the larger of two numbers in displays or pictures, yet still made errors at the completely symbolic level. The vocabulary of "is greater than" and "is less than" was mastered by K and he could verbalize statements such as "25 is greater than 19" quite well. The signs, < and > , were not mastered, and one possible reason for this is the perceptual difficulty he had in reversing digits (91 for 19, ∂ for 6). During an evaluation interview in February, K chose 56 as greater than 52 and wrote the sign (>) correctly. He also chose 27 as less than 41, but wrote the sign (<) incorrectly.

When three-digit numbers were introduced, and after not having recently worked with the concepts of more and less, K became very confused when asked to show on his abacus the number which was 1 more (less) or 10 more (less). For 10 more, he turned over 10 tens, but after three days of working with the "more" and "less" ideas, K began to feel confident that he could simply turn over 1 or 2 tens to show 10 more or 20 more.

Trading to show 1 or 10 more (less) was intriguing to K. He was the first in the group to show on his abacus the number which is 10 less than 506. Occasionally he traded incorrectly (1 hundred for 10 ones), but he was usually able to adjust his abacus correctly to show the number. One day after the group had been asked to show the number which was 1 less than 600, then 1 less than 300 and 1 less than 200, K became very excited. He exclaimed that he knew a short-cut for answering these questions: All the numbers would end with 99. His reason for this was that 1 less than 100 is 99, and so you just "put some hundreds in front of 99."

Worksheets which required counting by ones, tens and hundreds forward and backward were particularly frustrating for K. He was unable to write the numerals correctly unless he first showed the number on his abacus, yet he resented having to use the abacus since the other two boys were usually able to do their work without using it. K felt that the abacus slowed him down so as soon as he saw a pattern in writing the numerals, he quit using it. Then when he came to a number that required regrouping, he would put his head down and cry until the instructor would come over and ask him to show the number on the abacus. He would do so, then regroup correctly and go along merrily until he came to the next regrouping obstacle. Many times when filling blanks to show numbers which were ten more, K would get confused and begin to show one more or 100 more. He did not seem to be able to hold on to a visual pattern all the way along the line.

K completed the following to show ten more:

(205) 215 225 245 255 265 275 258 587

He completed the following to show one hundred less, ten less and one less:

210 310 410 510 (one hundred less)

390 490 500 510 (ten less)

OP3 OP4 OP5 510 (one less)

In the June evaluation interview, K was able to state confidently that 147 was 10 more than 137 because "after 3 comes 4." He also stated that 243 was 10 less than 253 because "5 take away 1 equals 4."

ADDITION AND SUBTRACTION WITHOUT REGROUPING

K was able to work addition and subtraction problems on the abacus with very little trouble. He was not as successful using pictures, however. When asked to find the picture which showed the sum of two addends on the instructor's abacus, K asked the instructor to join them first on the abacus so he could see how the answer would look. As long as all work was done at the enactive level, K was successful, but when he was asked to write a problem shown on an abacus or a picture, he had difficulty, primarily due to his writing and counting problems. When writing an addition problem, he counted one addend and wrote it, then the other addend and wrote it, the recounted the whole display and wrote the answer. It was possible for him to miscount on one or both addends and still have the correct answer. For example, for a picture of:

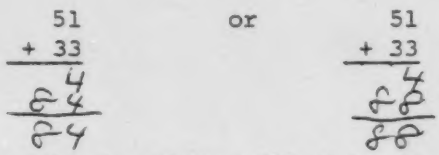
$$\begin{array}{r} 35 \\ +22 \\ \hline 57 \end{array}$$

K may have written: $\begin{array}{r} 53 \\ +22 \\ \hline 57 \end{array}$ or $\begin{array}{r} 35 \\ +21 \\ \hline 59 \end{array}$ or $\begin{array}{r} 35 \\ 22 \\ \hline 15 \end{array}$.

K also had difficulty in changing from counting by tens to counting by ones as mentioned earlier. When counting all the tens in a picture to write the sum, he frequently counted by tens when counting the tens for the bottom addend, then changed to counting by ones when counting the tens for the top addend.

K preferred to work with written problems only, or with the abacus only, and one could see why. At least when the problem was already written for him, he had less chance of making an error in counting the beads, or a reversal error in writing the numerals.

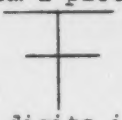
For addition problems, K had much difficulty at first writing the intermediate steps. He comes wrote;



After about a week of working with this, he suddenly seemed to understand where these partial sums came from and was then able to complete such problems without the aid of an abacus or picture.

Subtraction problems were more difficult for K to work with at the iconic level. When asked to find the picture which showed 5 - 3, he chose instead the picture for 8 - 3, which he interpreted as a set of 5 and a set of 3 being removed. K also had a tendency to reverse the take-away number when looking for a picture. For 57 - 31 he found a picture showing 57 - 13, and for 85 - 14 he found a picture for 85 - 41.

When writing problems from a picture, K many times set up a form



to help him write it. If the digits in a problem were not properly lined up, he was unable to correctly solve it. He was able to solve a problem such as

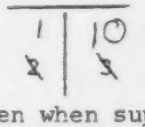
$$\begin{array}{r} 42 \\ +3 \\ \hline \end{array}$$

but became confused if the addends were reversed to show $\begin{array}{r} 3 \\ +42. \\ \hline \end{array}$

In an evaluation interview, K was given eight addition and subtraction problems to solve. He answered $\begin{array}{r} 5 \\ +23 \\ \hline \end{array}$ as 10, explaining that he added 5 and 3 and then 2 more. For the other seven problems he wrote correct answers with the exception of one counting error and one digit reversal. He was able mentally to compute other problems similar to these, however.

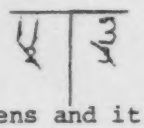
ADDITION AND SUBTRACTION WITH REGROUPING

K was able to go through the steps to show trading on the abacus, though he had some difficulties at the iconic and symbolic stages. When first learning to cross out digits and rewrite for a subtraction, he did it like this:



K put 10 over the ones column, even when supposedly using a picture which showed the trading sequence. This was cleared up soon, but another difficulty arose. After realizing that there were now 13 ones instead of

10 ones after trading, K wrote the digits this way:



He said that the 1 in 13 was for tens and it should go on the tens side.

K's reversal problems also caused computational problems, even after he had the algorithm mastered. The following problem points out this difficulty:

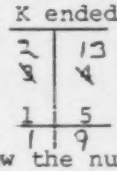
$$\begin{array}{r} 41 \\ +\ 9 \\ \hline 50 \end{array}$$

After addition with regrouping was introduced, K began using the long form for subtraction problems without regrouping. He worked problems in this manner:

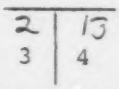
$$\begin{array}{r} 34. \\ -\ 21 \\ \hline 13 \end{array}$$

However, he did not write subtraction problems with regrouping in this manner. After about 5 weeks of working with addition and subtraction problems requiring regrouping, the group was given a "rule" for using the short form and long form. They were told that if the ones answer fit on the ones side (was a single digit) then the answer could be written in the short form. K was fascinated with this rule and immediately applied it to his work.

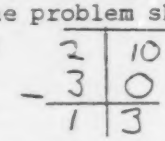
Pictures presented special difficulties for K. When a subtraction problem with regrouping was pictured, K ended up many times writing the following:



He would miscount the beads to show the number of ones after reading, then counted again to find the answer, which might be correctly counted. When K used symbols only, he would never have rewritten 34 as:



K also left out parts of a problem when asked to write the problem shown in the picture. For a picture which showed 30-17, he wrote:

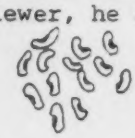


K had a great deal more difficulty writing subtraction problems from pictures than writing addition problems from pictures.

In an evaluation interview in April, K answered written problems correctly when he dealt only with symbols. However, he became confused when asked to show addition and subtraction problems with regrouping on the abacus and with beans. Of the four problems involving an aid, K finally answered 2 correctly after he went back to the symbols and adjusted the

manipulative display to model the symbols. For example, K showed 46 + 7 using beans like this:

He then put all the beans together, looked at the written problem and counted 6 + 7 on his fingers. After separating beans into 2 piles he pointed to each and said, "There's 13 here and 14 here," then said, I can't do this problem." After some questioning by the interviewer, he said the 4 beans were 40, and rearranged the beans like this:



This modeled the partial sums $\begin{array}{r} 13 \\ \underline{40} \end{array}$ he would have written for the problem. He finally said the answer was 53.

When asked to transfer his knowledge of addition and subtraction with regrouping to 3-digit numerals, K was somewhat successful. On the final evaluation interview, K correctly showed 3 of the 4 problems he was asked to work on the abacus. He could not work either of the 2 problems for which he was asked to use beans. The 2 written problems, where no manipulatives were used, he answered like this:

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline 5108 \end{array}$$

erased the 10 then wrote

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline 518 \end{array}$$

$$\begin{array}{r} 237 \\ -\ 176 \\ \hline 061 \end{array}$$

then erased the zero to leave 61.

IX. THE CASE OF L

BY

JUDY VORAN

BIOGRAPHICAL AND TESTING INFORMATION

L is a female child, born June 18, 1968. She lives with her mother, three sisters and one brother in a low rent housing project. L. is the middle child of the 5 children and 3 of them have different fathers, although there is now no father in the home. Her mother works as a maid, and the family income is very meager. L rarely talked about her family or her life outside school.

In the classroom L was usually quiet and somewhat withdrawn, although she did try to join in some small group activities occasionally. On a class sociogram L appeared as a loner and was chosen as a friend by few girls. However, L's self-concept was actually pretty good and she did not seem to be aware of her lower social or academic standing.

L did not attend kindergarten, and this was a detriment to her school performance. She attended first and second grade at the school at which the experiment was conducted and had the same teacher both years. In first grade she got off to a very slow start and the California Test of Basic Skills (CTBS), administered at grade 1.6, indicated this fact. L had a reading grade equivalent score of 0.2, and a language grade equivalent score and math grade equivalent score of 0.1. L participated in the county's Title I reading program and ESAA math and reading program, receiving small group remedial and corrective help throughout the year.

At the beginning of second grade, L had apparently forgotten most of what she had struggled to learn the year before. She was a total non-reader and could not count or recognize numerals greater than 6. However, during the year L received the Title I reading instruction and showed slow, steady improvement, though she remained in the lower third of the classroom reading groups. She was very slow learning to write letters and numerals and this writing problem was noticeable in all of her work. CTBS grade equivalent scores, administered at grade 2.6, were 1.8 in reading and 1.6 in math and in language. L scored at the 3rd percentile in the math computation subtest, which was a timed test, and at the 26th percentile on the math concepts and applications subtest, which was read by the teacher to the class as a whole.

On the PMDC second grade test administered in the fall, L was able to count sets by touching each object as she counted. She could not count forward beginning with a number other than one, nor could she count backward. She made no attempt to count by tens. L was unable to tell the number represented by bundles of straws and single straws or by colored chips, indicating complete lack of understanding of tens and ones. When instructed to use beans to find the answer to an addition problem L formed numerals with the beans, then counted on her fingers and told the answer. She used beans in a subtraction problem, but added instead of subtracting. For problems involving 2-digit numbers, L was unable to use the beans to solve them. L ordered one-digit numerals from smallest to largest, and correctly told which of two 2-digit numerals was more, though she could not tell which of two numbers was less.

On the spring administration of the PMDC test, L showed some improvement in her counting skills. She could count from 6 to 15, though not from 35 to 46. She counted backward beginning with 6, though she was unable to count back from 44. She counted by tens to 100, then said "101, 110, 120," but she was still unable to count by tens beginning with 26. In dealing with tens and ones, L was able to tell and write the number of a display of straws and of colored chips. She was able to construct a set to show a given number using colored chips, but not using straws, perhaps reflecting

L's instruction with the abacus during the year. L used beans to solve all addition and subtraction problems with the exception of one subtraction problem in which she added. L could choose from 2 numbers the number which is less and the number which is more, except for the pair (19, 31) when she chose 19 as being more.

On the KeyMath test given in the fall, L obtained a grade equivalent of 1.1. Relative strengths were the areas of money and mental computations (1 + 1, 2 + 2) and relative weaknesses were the areas of multiplication, division, missing elements and measurement. L answered one written addition problem correctly.

On the spring administration of the KeyMath test, L scored a 2.0 grade equivalent. Relative strengths were the areas of addition, subtraction and mental computation. Relative weaknesses were the areas of multiplication and missing elements. L answered 5 addition problems correctly, one of which involved regrouping, and 4 subtraction problems correctly, two of which involved regrouping.

As second grader in the T1 teaching experiment, L was assigned to the single embodiment group using the abacus (U3). There were 3 boys and 3 girls in this group, and L developed close ties with the other two girls. She relied heavily on them for answers and instructions initially, but after the first month of the experiment she became more independent, L worked slowly and determinedly throughout the year. She depended almost totally on the abacus to do all her work and rarely set it aside unless specifically directed to do so. Once she became accustomed to the symbolic patterns or manipulative strategies the group used, she showed steady improvement. However, any change in the schedule or absence from school affected her performance greatly since her retention was short.

The following discussion of L's performance in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

Counting past 10 was very difficult for L, and she was unable to read 2-digit numerals at the beginning of the year. She could count out 10 ones and trade them for 1 ten on the abacus, but usually arrived at the wrong display because she had miscounted when initially putting a given number of ones on the abacus. For example, L showed 16 as 1 ten and 5 ones because she had originally put 15 ones on the abacus and then traded 10 of them for a ten.

L learned to show a two-digit number on the abacus when she saw it in written form, but she had great difficulty showing it when the number was read to her. Hearing "34" meant little to her. She was unable to write it or show it, and she was unable to count by tens until December. Until then, she was not able to consistently count the beads on her abacus to tell what number was shown. She would begin counting the tens beads by tens,

but when moving to the ones rod, she continued to count by tens and was unable to bridge the gap (e.g., 30-40-41). She finally overcame this, somewhat, by counting the tens beads, then covering them with her left hand, then counting by ones on the ones rod. This covering up of the tens rod seemed to signal her to change her counting pattern.

L experienced difficulty on any new symbolic form when first introduced, but she soon picked up on visual cues, though not on the level of understanding. When first working with the following form, in October, she filled it in this way:

16

10 and 6

10 ten and 6 ones

When the form was revised and the lines ordered differently, L was much more successful:

2 tens and 6 ones

20 + 6

26

Worksheets which mixed up a pattern confused L, pointing out her lack of understanding of place value. One such worksheet had ____ tens = 30 and also ____ = 3 tens. She filled in both forms with the single digit (3). L also had a great deal of trouble verbalizing any of the patterns. When asked to tell a name for a number using tens and ones, she could rarely say "4 tens and 6 ones" without help, even when looking at the abacus. She could, however, answer the questions "How many tens?" and "How many ones?" When trying to tell a name for a number, she sometimes said "ten fours" instead of "four tens."

Moving into three-digit numbers in the spring magnified L's reading problems, though her writing problems remained much the same. L could show a three-digit number on the abacus and could trade when directed to get more or less tens or more or less ones. However, she could not read the numeral or tell the number from the abacus display. She said simply "5 hundreds and 6 tens and 2 ones." When asked to read a numeral, such as 431, she said "a hundred and 4 and thirty-one," and apparently thought that any number containing hundreds had to begin with "a hundred and . . . "

When trading for a display of 1 hundred 5 tens to show as tens, L meticulously counted out each of the 10 tens she put on the abacus. She was very slow to recognize quickly that 150 has 15 tens, 130 has 13 tens, etc., when working at the enactive and iconic levels. However, when she began completing worksheets with the form

15 tens

1 hundred 5 tens

she quickly saw the visual pattern. 150

When given an abacus picture showing hundreds and tens, L filled in the middle line of the form first, then the top line and finally the last line. When given an abacus picture showing tens only, L filled in the top line first, then the middle and bottom lines.

L could not answer a question such as "How many tens does 150 have?" but she could show 150 on the abacus as 1 hundred 5 tens, then trade to show 15 tens, then answer "15" to the question. L was unable to go through this process mentally without use of the abacus. During the final evaluation interview L was shown the statement 245 = 1 hundred 13 tens 15 ones and was asked if this statement was true. She quickly answered "No, because a 2 is supposed to be here (pointed to first blank) and a 4 is supposed to be here (pointing to the second blank) and a 5 is supposed to be here (pointing to the last blank)." When asked earlier in the interview to show

hundreds	tens	ones
2	13	4

on the abacus, she had done so correctly and had traded to show less tens, although she read the number only as "3 hundreds and 3 tens and 4 ones" and not as "three hundred thirty-four."

ORDERING

L experienced great difficulty in ordering numbers and in dealing with the concepts of more and less, due primarily to her inability to count and to read numerals. She was not able to count to 20 until December and made frequent counting errors throughout the year. When counting strategies were taught in a two-week fact drill unit, L never mastered the counting on strategy. If the problem was 7 + 5, she held up 7 fingers and counted each one "one, two, three . . . seven." Then she held up 5 fingers and counted "eight, nine, . . .twelve." She could not begin counting at 7, but needed to start with one each time.

Ordering pictures from 10 to 20 was an impossibility for L initially, since she did not know what number came after 14 or 18. Having to decide which picture to choose when more than two pictures were in front of her was more than she could manage. Her attention had to be focused on only two pictures at a time and then she had to decide which picture came before the other.

Reading a two-digit numeral, and some one-digit numerals, was a chore for L. In order to read 56, she had to count softly on her fingers "ten, twenty, thirty, forty, fifty" and only after that could she say "fifty-six." She went through this procedure every time she read a two-digit numeral through the end of the year. When asked to compare two written numerals, by the time she went through the process of reading the second numeral, she had forgotten what the first one was. L also reversed digits frequently, which caused ordering problems. When asked to place either < or > between 69 and 96, she said "Oh, no. That the same thing."

L judged the magnitude of a number by the height of the column of beads on the abacus without regard to tens and ones. Looking at pictures showing 7 and 43, she said 7 was larger. Even after locating both numerals on a hundred chart, she still insisted 7 was more than 43. In an evaluation interview in February she chose 56 as being more than 52 because one number had 6 and the other had 2. She then chose 41 as being less than 27 because "these (27) are higher," though she was looking only at the numerals and not at a picture or abacus display.

The vocabulary "is greater than" and "is less than" was never mastered by L. "Than," in particular, seemed a meaningless word. After working with the signs > and <, she was able to use them somewhat consistently, but could not read a statement she had written, such as 26>14. Even when word cards were used in place of the signs, she had to labor over the reading of the statement.

L could apply a rule given to her without real understanding of the concept, which is what she finally did to choose which of two numbers was larger or smaller. She was told that the number with more tens was the greater number, and if the numbers had the same amount of tens, then the one with the most ones was the greater number. It was apparent that she lacked a concept of more and less when she was asked to show a number that was 1 more, 10 more, 1 less and 10 less. Even in May she was not sure of whether to put another bead on the abacus or take one off when asked to show the number that was more. She would hesitate long enough to see what other children were doing, then would adjust her display accordingly. When asked to find the picture which showed 1 more (less) or 10 more (less) than a given number, she first had to show that number on her abacus, add or remove the appropriate bead, then look for a picture like her abacus. During the final evaluation interview L was asked, "What number is 10 more than 137?" She made no response to similar inquiries about 10 more than 37, 10 less than 253 and 10 less than 23.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

L was able to model addition and subtraction problems on the abacus, carry out the actual operation required and write the answer. She was only somewhat successful in writing the entire problem from an abacus or picture, and only rarely able to read the problem, either from the symbols, the picture or the abacus. L relied almost totally on the abacus to solve problems, even at the symbolic level, for more than half of the year. She knew she was able

to show two numbers on her abacus and then join them together, and she knew if she counted up the tens and ones and wrote that down, her answer would be correct, so she faithfully followed this process. L had no confidence in the symbols or in her ability to count on her fingers. She knew very few facts and did not have a good counting strategy to employ.

Pictures were difficult for L to interpret, especially for subtraction problems. For a picture showing 8 beads with 3 circled to show "take-away" L would write 5 - 3. When an addition problem was shown on the instructor's abacus and L was asked to select the picture which showed the answer to the problem, she could not do so unless she pulled on the rubber band holding up the top addend on the abacus, causing the two sets to be joined. When given a worksheet with pictured subtraction problems, L insisted on showing the number on her abacus, using a rubber band and carrying out the subtraction there first before writing the answer.

When using a picture to write an addition problem, L would count the tens beads at the top of the pictured abacus and write that digit down then count the ones beads and write that digit down. She repeated this process for the second addend at the bottom of the picture and drew the horizontal line and + sign. She then went back to the picture and counted all the tens beads and wrote the digit and then counted the ones beads and wrote that digit. She did not think of

$$\begin{array}{r} 36 \\ +12 \\ \hline \end{array}$$

as "thirty-six plus twelve equals forty-eight" but rather as "three and six then one and two, count and write four and eight." The same process was used for subtraction problems. This was probably due to her inability to read a two-digit number without first counting by tens until she said the correct decade.

L saw no need for the long form for addition problems, especially since she did not perceive adding of tens as such but as just adding of digits. When first working with the long form, she simply filled in the sum on every empty space. This was really the only information she got from a picture.

$$\begin{array}{r} 51 \\ +\ 33 \\ \hline 84 \\ 64 \\ \hline 84 \end{array} \qquad \text{or} \qquad \begin{array}{r} 51 \\ +\ 33 \\ \hline 4 \\ 64 \\ \hline 84 \end{array}$$

Gradually L began to deal more with the written symbols, and she began to use the abacus as a counter. For a problem such as 23 + 41, she would put on 3 ones, then 1 more one and count them and write down 4, then she would put on 2 tens and 4 more tens and count them and write down 6. When the abacus was taken away, she used her fingers for even the most obvious combinations, such as 9 + 0 or 6 - 1.

During an evaluation interview, L was given addition and subtraction problems without regrouping to solve. Although she was able to read only

3 of the 8 problems, she wrote the correct answer to every problem and, when asked, she worked the problems on the abacus correctly. However, when given problems of the same type ts solve mentally, she was unable to do any of them. When asked to show two problems using beans grouped as tens and ones, she was also unable to do so.

ADDITION AND SUBTRACTION WITH REGROUPING

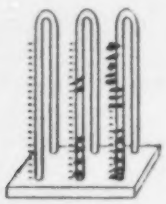
L encountered difficulty when learning to trade to have more ones, needed for subtraction with regrouping, and to have less ones, needed for addition with regrouping. Some problems arose from counting errors, but other problems indicated a lack of understanding of the whole regrouping process. L could not remember how many ones to trade for a ten, even though the name of the bead ("ten") should have helped her remember. When first trading ones for a ten, she would trade all the ones above the 10 mark on the abacus. For example, if the abacus had 3 tens and 16 ones on it, L would pick up the top 6 ones, having 10 ones on the abacus, and trade the 6 ones for a ten. She sometimes picked up a ten from the 3 tens on the abacus, then put it back with the other 2 tens to show trading for the ones, instead of bringing over a ten from the back. L did not seem to realize that a trade involved moving something from the front of the abacus to the back <u>and</u> moving something else from the back of the abacus to the front.

L counted on to cross out numerals to show regrouping necessary for subtraction problems. For

3	4

L first crossed out the 4, then held up 10 fingers and counted on from 4 to get 14, then wrote 14 above the 4. She then crossed out the 3, held up 3 fingers and took one away, and could immediately write 14 above a 4, 16 above a 6, etc. She did not know that 10 + 4 is 14, 10 +6 is 16, etc., unless she counted on her fingers, and she could not trade a ten on the abacus to show 14 ones unless she counted out each of the 10 ones she added. The symbolic pattern she perceived was purely symbolic, with little relation to the enactive level.

When pictures were involved, they brought their own special difficulties to L. She could count the total number of beads to find the answer to an addition problem with regrouping, but she could not relate the picture to the structured situation where she wrote partial sums. For example, L responded this way to the following situation:



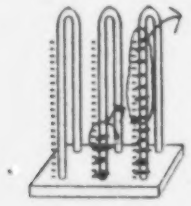
4	6			4	6	
+ 2	8	or		+ 2	8	
			ones in all	4	6	ones in all
7	4		tens in all	2	8	tens in all
7	4		altogether	7	4	altogether

At times L crossed out to show trading for subtraction problems when it was not necessary to trade. She would write:

~~2~~	14
~~3~~	~~4~~
- 1	1
2	3

and the answer would be correct because she counted it from the picture, although she went through the regrouping steps without looking at the picture.

When L was asked to write the entire problem from a picture, she had the greatest difficulty translating a subtraction situation to written form. For:



she wrote
$$\begin{array}{r} 28 \\ -\ 25 \\ \hline 53 \end{array}$$

L usually correctly wrote a pictured addition situation, but when the structure for partial sums was not present, she wrote addition problems without regrouping the short form (e.g., $\begin{array}{r} 32 \\ +14 \\ \hline 46 \end{array}$) and addition problems with regrouping in this form:
$$\begin{array}{r} 24 \\ +\ 18 \\ \hline 12 \\ \hline 42 \end{array}$$

A new kind of error surfaced when L began working only with the symbols. For a problem such as:

$$\begin{array}{r} 34 \\ -\ 16 \\ \hline \end{array}$$

L held up 10 fingers, took away 6, then wrote 4 in the ones column. After being told that she must do trading first and write down the numerals above the crossed out numerals, she did not correctly. On problems such as

3	4
- 1	2

L sometimes wrote the answer this way:

~~2~~	~~2~~
3	4
- 1	2 .

Evidently she felt it was necessary to fill up the "regrouping' space when no regrouping was needed.

During the final evaluation interview L was asked to solve 3 digit addition and subtraction problems which required regrouping. She correctly displayed addends on the abacus and with beans and could join them, but she did not trade correctly on addition problems. When asked to work the following problem on a symbolic level only, she did so like this:

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline 508 \end{array}$$

She could not correctly show subtraction problems using the abacus and beans. When asked to work the following problem on a symbolic level only, she did so like this:

$$\begin{array}{r} 13 \\ 237 \\ -\ 176 \\ \hline 161 \end{array}$$

She was unable to read any of the problems, now could she tell the answer she had on the abacus, with beans or written down.

X. THE CASE OF N

by

PATRICIA CAMPBELL

BIOGRAPHICAL AND TESTING INFORMATION

N is a female child, born February 13, 1968. She lives with both of her parents, one sister, four years older, and a baby brother. During the late fall, N learned that her mother would have a baby in the spring. N wanted the baby to be a girl and adamantly stated that, if the baby was a boy, she wanted it sent back. However, by the time her baby brother was born, N accepted him saying that "God wanted" her family to have a boy.

N's parents are high school graduates. N's mother does not work outside the home. N's father was a salesman throughout most of the school year; however, for a time in the spring he was unemployed. N's family had a lower middle class income and formed most of their social contacts through their church.

N did not go to kindergarten, rather she attended a day care center for 6 months before her family moved to this city and N entered first grade at the school at which the experiment was conducted. N was a good student in first grade, but not exceptional. She scored slightly above grade level on the Comprehensive Test of Basic Skills (CTBS) administered at grade 1.6 with an overall grade equivalent of 1.8 (reading, 1.8; language, 1.8; mathematics,2.1).

At the end of the first grade, N's father and older sister began to teach her at home. This continued throughout the second grade, as N and her sister would "play school."

During second grade, N became the top student in her class and, during midyear, was identified as a gifted student eligible for special programs sponsored by the school district. Her entering second grade Otis-Lennon IQ was 124. On the CTBS administered at grade 2.6, N had an overall grade equivalent of 5.1, with reading scales of 4.5, language scales of 5.9, and mathematics scales of 4.5. N liked to be challenged and learn new things, but she was also very understanding toward children who did not learn as quickly as she did. N was popular in the classroom, although somewhat bossy in her independence and self-confidence.

N had an entering KeyMath grade equivalent of 3.0. Her counting skills were advanced as she could identify missing numbers in a written sequence (____, 6, 7, ____, 9; 98, 99 ____ 101) and could compare the numerousness of two given sets. However, N could not recognize counting by 3's. N could add and subtract without regrouping and solve missing addend problems. She could perform mental computation problems of sums less than 10. If an accurate picture prompt was present, N could solve word problems presented orally; she was not successful if the picture prompt was absent or inaccurate.

In May of the second grade, N was again administered the KeyMath test, receiving a grade equivalent of 3.8. At this time, N succeeded on additional items. She could fill in missing numbers of sequences representing counting by 4's. She could solve addition and subtraction problems with regrouping involving two-digit numbers, but not with three-digit numbers. N could also solve more complex mental computation problems involving two operations (e.g., 5 + 5 − 4 + 7) and she could multiply single-digit numbers. N could solve orally presented word problems without picture prompts, but erred if an inaccurate picture prompt was present.

On the PMDC second grade test in the fall, N could count forwards by tens to 130, but when counting by tens from 26 to 126, she skipped 106 and 116. N counted sets of objects visually. Given bundles of ten straws or colored chips to represent tens and ones, N could count by tens and ones to determine the value of the display or to form a set depicting a particular number. She could use counters to solve an addition or subtraction problem, but she occasionally arrived at the wrong solution due to miscounting (e.g., 23 - 7 + 15). N could identify which number was more or less in a set of two numbers, $\{7, 4\}$ and $\{8, 12\}$; but she thought that 19 was more than 31.

During the spring of the second grade, N again took the PMDC second grade test. She was successful on all tasks, with the exception of one class inclusion item.

As a second grader in the T1 teaching experiment, N was assigned to the multi-embodiment group (M) using sticks, Dienes blocks, and the abacus.

Of the six children orginally composing this group, N was usually the first to successfully complete written symbolic exercises. This was not apparent during lessons involving the embodiments lessons or involving pictures of the embodiments without written work. During the introduction of written symbolic strategies, N would use the materials to complete the exercises. Later, once she understood how to solve the problems symbolically, she used the manipulatives only as a means to check her work, or if requested to explain her work to the teacher.

For the last five months of the teaching experiment, N was the only female student in the M group. This had no noticeable influence on her performance.

The following discussion of N's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

Counting and forming displays with the embodiments to represent numbers less than 100 was a simple task for N. At first, when she was forming a display and hunting sufficient sticks or units, N would lose her place in the counting sequence and start counting over again. She quickly learned to recall the last number she had counted and, after locating more sticks or units, count on from that point. N also was one of the first students to adjust a previous display in order to show a subsequent value (e.g., add 2 sticks to 17 sticks previously counted in order to show 19 sticks). When displaying larger values (greater than 20), N would sometimes ask her instructor to request "sticks again" or "blocks again" rather than changing the embodiment so that she could simply adjust her display.

N had no difficulty in changing from one embodiment to another and could easily represent a number pictured or displayed in one manipulative by one of the other manipulatives. During the third week of instruction, in order to make her displays different from those of the other children, N began interchanging the embodiments. For example, she would represent 46 with 4 bundles and 6 units or 32 with 3 tens on the abacus and 2 sticks. By this time all of the children in the M group were proficient with the vocabulary and this generated discussion as to whether N's displays were correct. The general concensus was that units meant units and only units, similarly for longs or bundles or sticks (e.g., "Show me 3 longs and 2 units . . .How many is that?"). But there could be many correct ways of showing tens and ones or of simply showing a number. After a week or so, N seemed to tire of interchanging the embodiments and stopped doing so.

N had no difficulty reading or writing numbers or reading the terms: units, longs, tens, ones, bundles, sticks. She easily filled in worksheets

containing exercises such as:

3 bundles and 6 sticks

3 tens and 6 ones

30 and 6 ones

36.

Given a two-digit number, N could explain the meaning of the digits verbally in terms of tens and ones or she could use manipulatives to explain the numerals.

In the spring of the year, N adapted to three-digit numbers between 100 and 200 with similar ease. She quickly learned to describe a number (written or spoken) or a display in terms of hundreds, tens and ones. At first she was hesitant and not particularly accurate at determining how many tens were in a number such as 170 or 190 but she was more confident with numbers near one hundred such as 120 or 130. Similarly, she was sure 12 tens was 120 or 11 tens was 110, but she was hesitant to suggest what number was 18 tens. All of this seemed to crystallize for N when she ordered pictures depicting 10 longs (100), 11 longs (110), 12 longs (120), . . . 20 longs (200) and pictures depicting 1 bundle of bundles (100), 1 bundle of bundles and 1 bundle (110), 1 bundle of bundles and 2 bundles (120), . . . 2 bundles of bundles (200). At first, N occasionally would err when counting displays of blocks (longs and flats) by tens, (10, 20, 30, . . . 90, 100, 10, 20), however this problem also seemed to clear up after ordering the pictures. N completed response sheet exercises such as:

15 tens

1 hundred 5 tens

150

with little difficulty. During group work, these exercises were prompted in differing ways. Initially a display was made to depict each line, then later only one line was prompted by a display and the students completed the exercise. After a time the embodiments were not used. The instructor would fill in one line of the exercise, and the students would complete it. N became bored with these activities as she said they were all the same and she wanted to do something different.

Numbers greater than 200 did not present too much difficulty for N. She initially thought 3 bundles of bundles (300) would be made of 13 bundles. However, after the bundle of bundles were taken apart and the group started to count the bundles, she stated, "I mean 30." On the next item with blocks, she correctly stated that 8 flats (800) would be the same as 80 longs. N could describe a given number less than 1,000 in terms of hundreds, tens and ones or in terms of tens and ones. She could change a display depicting a number less than 1,000 involving any of the

embodiments so that it would show the same number but with "more tens" or "more hundreds." Nevertheless, N may not have had a real sense of the larger numbers as when asked to tell a number between 500 and 600, she could not do so until told that it would sound like "five hundre . . . . .".

At the end of the school year, N had a concept of place value that she could transfer to new situations. During her final interview N was shown the following statement and asked if it was true:

245 = 1 hundred 13 tens 15 ones.

She had not seen statements of this type during her instruction. N stated that the expression was true, "'cause that's really 100 (pointing to 13 tens) so that makes 200. And 3 tens plus 1 ten from that one (pointing to 15 ones) makes 4 tens and that (pointing to the 5 of 15 ones) makes 5."

ORDERING

Ordering pictures depicting numbers from 10 to 20 involving different manipulatives was initially challenging for N. Larger numbers also were more difficult for when N was ordering block pictures depicting numbers from 80 to 90 she said with some surprise, "Ms. C., this is hard!" Also increasing the number of pictures to be ordered increased the difficulty of the task for N.

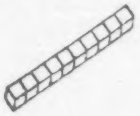
N could tell which of two displays or pictures was "more, less, greater, or fewer." She would determine the numbers represented by the displays or pictures and decide more or less on the basis of the numbers, not on the basis of the numerousness depicted by the pictures or displays. She quickly adapted to using the formal statement "___ is greater than ___" or "___ is less than ___," in both written and a verbal form, as well as placing the symbols "$<$" and "$>$." Shown a set in one embodiment, she could easily form a display that was more or less than the stimulus set. N seemed to wait to see what the other children would do and then make a display that was correct, but different in appearance from the other children's. For example, when asked to make a set that was more than 27 (shown by blocks), N put out 10 longs. When asked to make a set that was less than 13 (shown by sticks), N did nothing and said that her set showed 0.

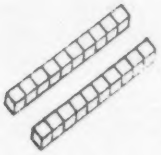
After showing a value with the manipulatives if asked to show a number that was 1 more (less), 10 more (less), 2 more (less), etc., N would first state what the adjusted number would be and then change her display to reflect that number. N also revealed her grasp of ordering at a more abstract level when she explained that 27 was less than 41 because "4 tens is more than 2 tens" or that the number that was 10 more than 137 was 147 "because 3 plus 1 is 4 and 30 plus 10 is 40."

ADDITION AND SUBTRACTION WITHOUT REGROUPING

N was able to use the embodiments to show both addition and subtraction. She readily agreed that displays depicting a number sentence could be made in

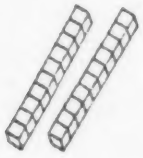
either a horizontal or a vertical form; however, when writing an additive number sentence to represent a pictured display she imitated the display. For the pictures:





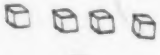
N wrote $\begin{array}{r} 10 \\ +\,20 \\ \hline 30 \end{array}$

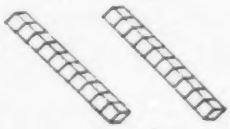
But for



N wrote $10 + 20 = 30$

The only exception to this was when the pictures reflected both two-digit and one-digit numbers in a vertical form. N always wrote the two-digit number on top:

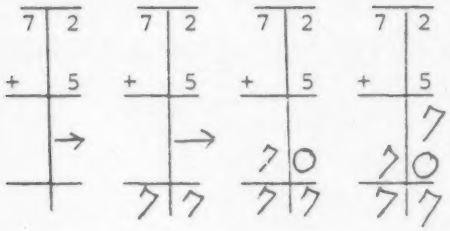


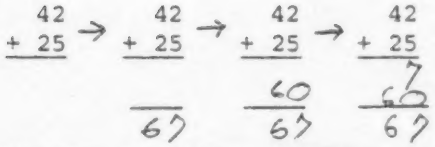


$\begin{array}{r} 20 \\ +\,5 \\ \hline 25 \end{array}$

When adding without regrouping using either sticks or blocks, N would keep the tens and ones separated on the form board; she would then add the tens and ones simultaneously using both hands to sweep the tens together and the ones together. Symbolically, she sometimes added the tens first and sometimes added the ones first; she often subtracted the tens first. When told to order pictures to show an addition problem without regrouping (e.g., 21 + 13), N would first locate the pictures depicting the two addends. Viewing the two pictured addends, she would almost immediately know the solution and then search for the picture which depicted that number.

Although N preferred to use the short form for addition problems, she would show her work symbolically on the long form. However, she filled in her work after solving the problem. This occurred for problems prompted by manipulative displays or pictures as well as on symbolic problems with and without the place value chart structure.





On the computation problems, N always wrote a + or - sign and lined up the digits carefully. She infrequently erred on symbolic computation problems by performing the wrong operation. N knew the basic facts for addition and subtraction (e.g., 5 - 3 = 2) and could transfer that knowledge to mental computation problems involving larger numbers (e.g., 50 - 30 = 20). When solving involved mental computation problems, she seemed to function at an abstract level, almost picturing the symbols in her mind. For example, when asked to solve "43 take away 32" mentally, she said, "11 . . . .(Because) 4 take away 3 equals 1 and 3 take away 2 equals 1 so you have 11 on the bottom."

ADDITION AND SUBTRACTION WITH REGROUPING

N had no trouble adjusting to solving addition and subtraction problems with regrouping. After a few enactive lessons on addition with regrouping and again after a few enactive lessons on subtraction with regrouping, N asked if they were learning to "borrow" and to "carry". She was quite pleased when she was told that was a name given to the work they were doing.

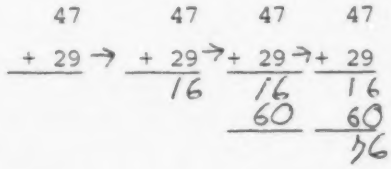
N was quite proficient at using all three embodiments to solve addition and subtraction problems with regrouping and to explain her solutions. For example, when solving the problem:

$$\begin{array}{r} 58 \\ +\ 25 \\ \hline \end{array}$$

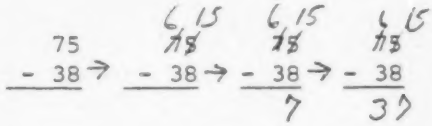
with longs and units, N displayed the two addends and jointed the tens and ones; she then had 7 longs and 13 units. N explained, "There's 7 from the tens altogether and there's one more ten over here (Motions toward the 13 units. N then traded 10 of the units for 1 long.). And you would have 3 more left if you put the 1 ten here (places the 1 long with the 7 longs)." The answer, she stated, was 83.

Similarly, N handled addition and subtraction problems with regrouping at the symbolic level with little difficulty. Her errors were due to fact

errors (e.g., 8 + 6 = 13) rather than a misunderstanding of the concept. N always added the ones and wrote that solution before adding the tens.:



Similarly, she always subtracted the ones first and, if she needed to regroup, noted it symbolically. For example, N worked the following subtraction problem:



Explaining her work, N said "You can't take 8 from 5 so you trade. You mark out the 5 and put 15; mark out the 7 and put 6. 15 take away 8 is 7 and 6 take away 3 is 3 . . .(The answer is) 37."

N was able to use her understanding of the concept of regrouping in new situations. At the final interview, N was asked to solve addition and subtraction problems with regrouping involving three-digit numbers. She was not instructed how to do it, but was allowed to use the manipulative of her choice. N selected the abacus and correctly solved the problems. She worked these problems from left to right, starting with the hundreds. For example, N solved:

$$\begin{array}{r} 156 \\ +\ 178 \\ \hline \end{array}$$

on the abacus in the following manner:

1. Displayed 156 as 1 hundred, 5 tens, 6 ones.
2. Placed clips on the abacus above the 156.
3. Displayed 178 as 1 hundred, 7 tens, 8 ones.
4. Added the hundreds by removing the clip.
5. Added the tens by removing the clip.
6. Traded 10 tens for 1 hundred.
7. Added the ones by removing the clip.
8. Traded 10 ones for 1 ten.
9. Stated the answer was 334.

Similarly for the problem:

$$\begin{array}{r} 524 \\ -\ 183 \\ \hline \end{array}$$

N used the abacus as follows:

1. Displayed 524 as 5 hundreds, 2 tens, 4 ones.
2. Took away 1 hundred.
3. Traded 1 hundred for 10 tens.
4. Took away 8 tens.
5. Took away 3 ones.
6. Stated the answer was 341.

After solving four problems of this type on the abacus, N solved three-digit addition and subtraction problems with regrouping symbolically. Again she worked from left to right:

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline \end{array} \rightarrow \begin{array}{r} 426 \\ +\ 182 \\ \hline 5 \end{array} \rightarrow \text{(erases 5)} \begin{array}{r} 426 \\ +\ 182 \\ \hline 100 \end{array} \rightarrow \text{(erases 100)} \begin{array}{r} 426 \\ +\ 182 \\ \hline 608 \end{array}$$

N explained, "400 plus 100 is 500. 8 tens plus 20 is 100. 500 plus 100 is 600 and 6 plus 2 is 8."

Similarly for subtraction:

$$\begin{array}{r} 237 \\ -\ 176 \\ \hline \end{array} \rightarrow \begin{array}{r} {}^{1}\not{2}\not{3}^{13}7 \\ -\ 176 \\ \hline \end{array} \rightarrow \begin{array}{r} {}^{1}\not{2}\not{3}^{13}7 \\ -\ 176 \\ \hline 61 \end{array}$$

Again, N explained her work: "Well, you can't take away 7 from 3 so I traded. One ten and the 3, that's 13. And 1 take away 1 is none. And 13 take away 7 is 6 and 7 take away 6 is 1."

XI. THE CASE OF P

BY

PATRICIA CAMPBELL

BIOGRAPHICAL AND TESTING INFORMATION

P is a male child, born February 16, 1968. He is the youngest of four children with two older brothers, ages 29 and 19, and an older sister, age 15. P and his teenage brother and sister have a somewhat unstable home life, as they live either with both parents or, at times, some of the children may reside with only one parent. P's mother is a high school graduate; she does not work outside the home. His father is a manual laborer; he did not graduate from high school.

P is an aggressive child who is very defensive. He has no sense of humor and frequently interprets innocent statements by other children as an insult causing him to strike back physically, in defense of his character. P cannot stand to be touched or have his materials handled by the other children. A sociogram of his second grade class showed that P was either named as a best friend (by students who were somewhat of a behavior problem in the class) or he was not named as a friend. P has no children near his age to play with outside of school; they are all older than he is. P's mother reported that at home, P is often belittled by the older children. At school, P attempts to imitate them.

P attended kindergarten at the school at which the experiment was conducted. During the first and second grade, P was in the Title I Program. P did not like to do group work, but preferred individual tasks. He was conscientious about his own work and corrected his errors readily, unless he had an audience of other children to entertain with his antics. He seldom would ask for help even if he was confused. Only when he felt that the other children were preoccupied with their own work, would P ask for assistance. P had an entering second grade Otis-Lennon IQ of 96 but, in the classroom, P was handicapped by his inability to read. On the Comprehensive Test of Basic Skills (CTBS) administered at grade 2.6, P had a total reading scale grade equivalent of 1.2 with a language grade equivalent of .9, placing him in the lower 5 percentile range. His mathematics scale on the CTBS was 2.1, with a total overall grade equivalent of 1.6.

On the KeyMath test administered at the beginning of second grade, P had an overall grade equivalent of 1.8. He could count and identify missing numbers in the sequence ___, 6, 7, ___, 9 as well as tell the number which preceded 19. However, he could not identify the missing number in the sequence 98, 99, ___, 101 or compare the numerousness of two given sets. P was able to solve addition and subtraction problems of sums less than 10, but he did not attempt any computation problems involving two-digit numbers. P could solve mental computation problems involving one, but not two, operations; he could not solve missing addend problems. P was not successful in solving word problems presented orally; even if the picture prompts were available.

In the spring of the second grade, P was again administered the KeyMath test, receiving an overall grade equivalent of 2.4. He still could not compare the numerousness of two sets, but he was able to identify the missing number in the sequence 98, 99, ____, 101. P successfully solved addition and subtraction problems involving two-digit numbers with and without regrouping; however, he occasionally erred due to performing the wrong operation. He also could successfully perform mental computations involving two operations if the sums were less than 10 (erred on 5 + 7 - 3). If accurate picture prompts were present, P was able to solve orally presented word problems.

On the second grade PMDC test in the fall, P counted by touching objects. He could count forwards from 6 to 15 and from 35 to 46 but he could not count back from 44 to 25. When counting by tens, P counted 10, 20, and then usually continued by ones as either 10, 20, 21, 22, . . .

or 10, 20, 23, 24, 25, . . . Therefore, he was not able to successfully solve place value tasks involving chips. P knew how to use counters to solve addition problems such as 18 + 5 and 2 + 4. However, he used the counters to add rather than subtract when presented the problem 7 - 3; he would not attempt 23 - 7. P could identify which number was more or less from the sets $\{7, 4\}$ and $\{8, 12\}$, but felt 19 was more than 31.

In the spring, P again took the PMDC second grade test. At that time he could count backwards as well as count by tens from 10 to 130. When he counted by tens from 26 to 126, he skipped 116. P was able to solve all place value tasks correctly and to identify numbers which were more or less. With the counters, P again added for all the problems, this time stating that 23 - 7 was 30.

As a second grader in the T1 teaching experiment, P was assigned to the multi-embodiment group (M) using sticks, Dienes blocks, and the abacus. When solving problems at the symbolic level, P would sometimes use his abacus or his blocks as an aid in beginning the exercises. However, once he understood what he was to do in the exercise, he would stop using the manipulatives as an aid. P was frequently one of the last to complete written exercises, but it did not seem to bother him.

The following discussion of P's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regouping, addition and subtraction with regrouping.

PLACE VALUE

P had no difficulty using the different manipulatives. At first when making displays with sticks or blocks to represent numbers greater than 15, he would lose his place in the counting sequence while hunting for sufficient sticks or units and would have to start counting over again. However, he gradually learned to recall the last number counted and count on from that point. Sometimes, P would adjust his previous display to show a subsequent number, but he usually did not. P had no trouble forming displays using one manipulative which were prompted by displays or pictures depicting the other embodiments. He easily moved from one embodiment to another.

Written exercises of the form:

___ long and ___ units

____tens and ___ones

____and ____ones

_____

were very difficult for P since he could not read the words. He was able to verbally describe a manipulative display as, for example, 2 longs and 3 units or as 2 tens and 3 ones. If asked what number the display represented, he would respond 23. However, characterizing the display as "20 and 3 ones" was not natural to P and this caused some confusion as he began writing and saying "____ tens and ____ ones."

Despite his awkwardness in describing pictures or objects orally or in written form, P had no difficulty using the manipulatives to show a number which was orally described in terms of either bundles or sticks, tens and ones, longs and units, or ____ and ____ ones.

At the beginning of the T1 teaching experiment, P could not count by tens. He learned to translate from the number of tens to the number, counting 1 ten, 2 tens, 3 tens, 4 tens, . . . 40," eventually becoming proficient at counting by tens. Later, P had difficulty counting by tens and ones. Particularly with the abacus, P would repeat counting by tens over and over again before counting by ones. For example, he would count:

"10, 20, 30, 40, . . . 10, 20, 30, 40, . . . 10, 20, 30, 40,. . ."

before finally saying: "10, 20, 30, 40, 41, 42, 43, 44, 45."

P wrote two-digit numbers greater than 20 by translating from tens and ones. That is, if P had to write the number "fifty-seven," he would say "5 tens, 7 ones" and write "57." Occasionally he erred when writing the numbers 16 and 17, writing the digit in reverse order as 61 or 71. Given a written number, P easily explained or displayed the value of the digits. Although P decided how to write a number by interpreting it as tens and ones, he seemed to be able to read numbers without verbally translating the digits in terms of tens and ones. This was also the case later in the year with three-digit numbers as P would interpret a number in terms of hundreds, tens, and ones before writing it, but he would read the three-digit number without initially interpreting its place value.

P quickly learned to describe a number or display (less than 200) in terms of hundreds, tens, and ones. Although P was certain that 100 was 10 tens and conversely, that 10 tens was 100, he had no idea what number was, for example, 17 tens or 19 tens. He was more comfortable with numbers near 100 as he would suggest that 12 tens may be 120, but he also wanted to check his conjecture on the manipulatives before stating it with any conviction. All of this seemed to suddenly make sense to P when he ordered pictures depicting 10 longs (100), 11 longs (110), . . . 20 longs (200) and pictures depicting 1 bundle of bundles (100), 1 bundle of bundles and 1 bundle (110), 1 bundle of bundles and 2 bundles (120), . . . 2 bundles of bundles (200). After that lesson P could tell how many tens were represented in any multiple of ten between 100 and 200 and, conversely, he could state the number for any number of tens between 10 and 20.

P could describe a number greater than 200 or a manipulative display representing a number greater than 200 in terms of hundreds, tens, and ones. He could use the manipulatives to represent such a number and could also adjust his display so that it would represent the same number, but with "more tens" and "more hundreds." Told a number of tens (e.g., 27 tens), P soon became proficient at displaying the number or finding the picture that depicted the value. Once he had found the picture or formed his manipulative display to represent the tens, P would count to determine the number represented. If asked how many tens were represented by a number greater than 200, P could use his manipulatives to find the answer.

However, P could not solve these problems without the manipulatives and he had difficulty stating his solutions as if the process of verbalizing was hampering him.

E: P, what number would you have if you had 90 tens?
P: 190

E: Think again. 90 tens. 90 of these things (points to a pile of longs).
P: I have 90 tens?

E: (Nods her head affirmatively). You have 90 of these (E picks up some longs). How many units would you have?
P: 100 and . . . (no further response).

E: Well, let's check it out. We want to have 90 tens.
(P and E count 90 longs, placing them in groups of 10 longs. P lines up the groups of the longs so they each resemble a flat).

E: Now, you've got 90 tens. Now here's a hint. (E picks up a flat moving it from one group of longs to the next showing that the flat "fits" exactly over the group of 10 longs.)

E: How much have you got?
P: (As he touches each group of 10 longs) 100, 200, 300, 400, 500, 600, 700, 800, . . . 900.

E: 900. Okay . . . So 900 is how many tens?
P: 9 . . .190 . . . 90, 9, . . .

E: What?
P: 9, 109, . . . mmm, 119, . . . 9.

When counting by ones or tens, P usually hesitated before bridging the hundred. He may not have had a real sense of the larger numbers for when asked to tell a number between 500 and 600, P responded, "287 + 369."

P initially had some difficulty on the written response sheet exercises as he repeatedly answered in the following manner:

17 tens

100 hundreds 7 tens

170

However, on worksheet exercises, P used the pictures as an aid and correctly answered problems of this type.

P seemed to have acquired the concept of place value by the end of the second grade. He often decided how to write a number by translating it in

terms of tens and ones or hundreds, tens, and ones. He also was able to apply his concept of place value in a new situation. During the final interview, P was shown the following statement:

245 = __1__ hundr__13__ tens __15__ ones

He had not seen statements of this type during his instruction. P stated the expression was true " 'cause you got 13 tens and if you trade . . . you get 200. Then 3 tens. Trade. Get 2 hundred . . . forty . . . five."

ORDERING

Ordering pictures depicting numbers from 10 to 20 involving differing manipulatives was very difficult for P, particularly when dealing with abacus pictures. Similarly, throughout the year, ordering pictures depicting larger numbers or ordering pictures representing non-consecutive numbers was a very challenging task for P but, with sufficient time, he usually was successful.

P could tell which of two displays or pictures was "more, less, fewer, or greater." He seemed to initially compare the numerousness of the sets and then check his answer by counting. Shown a set in one embodiment, P could easily form a display that was more or less than the stimulus set. For example, when asked to make a set that was more than 27 (shown by blocks), P displayed 4 longs and 5 units. P never did easily use the formal terms "____ is greater than ____" or "___ is less than ___." These phrases seemed to hinder P as he would concentrate so much on the form that he would forget the value of one of the displays or pictures before he completed the phrase or he would say, for example, "92 is better than 29."

At the symbolic level, P could not read the words "is" or "than" with any consistency; hence, he had difficulty placing the "is greater than" or "is less than" cards between numeral cards. P seemed to place "$<$" or "$>$" cards readily enough, but hesitated to read either the symbolic or the written statements. Usually, P would repeat the numbers involved aloud and then make up either an "is greater than," an "is less than," or an "is better than" statement. Later in the year, when asked to read the statement:

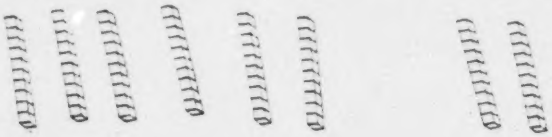
$$56 > 52$$

P replied, "56 is higher than 52."

After showing one number with the manipulatives, if asked to show a number that was 1 more (less), 2 more (less), 10 more (less), 100 more (less), etc., P would adjust his display and then count before stating the adjusted number. The only exception to this was with the abacus. If asked the number that was 1 more (less), 10 more (less), or 100 more (less) than the value displayed on the abacus, P could answer correctly without adjusting the display. However, if asked the number that was, for example, 2 less or 20 more than the value displayed on the abacus, he would adjust the display and count before responding.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

P was able to use the embodiments to show both addition and subtraction problems without regrouping. He readily agreed that either horizontal or vertical displays with differing embodiments could be used to depict the same number sentence. When writing a number sentence for a pictured display, for example:

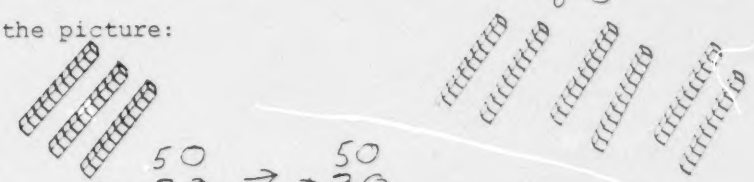


P would horizontally write: 60 + 20 = 80

But vertically he would "write up" in the following fashion:

$$+ \rightarrow +60 \rightarrow \overset{20}{+60} \rightarrow \begin{array}{r} +60 \\ \hline 80 \end{array}$$

Similarly, for the picture:



P wrote:

$$\pm 30 \rightarrow \overset{50}{\pm 30} \rightarrow \begin{array}{r} 50 \\ +\ 30 \\ \hline 80 \end{array}$$

This procedure of "writing up" the problem continued throughout December. After Christmas vacation, P stopped "writing up" a problem.

When adding without regrouping using either sticks or blocks, P would keep the tens and ones separated on the form board; he would then add the tens and ones simultaneously using both hands to sweep the tens together and the ones together. Symbolically, P added or subtracted the tens before adding or subtracting the ones. When told to order pictures to depict an addition problem without regrouping (e.g., 21 + 13), P would first find the pictures to show each addend. Then he would count the total number of tens and the total number of ones depicted on the two pictures (3 tens, 4 ones). Finally he would look for the picture that depicted that number of tens and ones.

P initially had difficulty using the longer form for addition problems. For the problem:

3	3
+ 2	2
	5
5	0
5	5

P wrote

3	3
+ 2	2
50	5
5	5

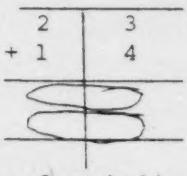
Later he wrote:

10	14
3	0
4	5

rather than

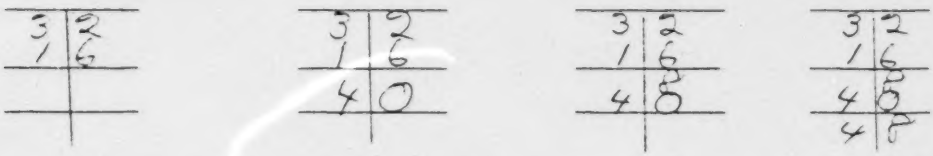
1	4
+ 3	1
	5
4	0
4	5

Ovals were then added to the structure of P's worksheets, and he then completed the exercises successfully.



On the next set of symbolic worksheets, P drew ovals before attempting to solve the problems.

Further use of the manipulatives on the form board in conjunction with the actual writing the problem used as a record of his actions seemed to be very beneficial for P. He was able to explain the meaning of his written work with the manipulatives. At this time, P began adding the tens first, with the manipulatives, and also recording in that order.



P did not want to use his fingers to solve the basic facts involved in the symbolic addition and subtraction problems without regrouping. He would count on to add and count back to subtract. Initially this led to errors as, for example to solve:

$$\begin{array}{r} 48 \\ -\ 25 \\ \hline 24 \end{array}$$

P would count back "8, 7, 6, 5, 4" and write "24" as the solution. Similarly, he would count on to add 8 + 6 by saying, "8, 9, 10, 11, 12, 13," and write 13 as the solution. After instruction with the embodiments, P learned to count on and count back with accuracy. However, he did not pay much attention to the operation signs. Therefore, he would often add when he should have subtracted and subtract when he should have added.

During oral drill, P seemed to understand the relationship between the basic facts for addition and subtraction such as "8 take away 3" and the similar statements, "8 tens take away 3 tens." However, he could not transfer these solutions to problems such as 80 - 30. Yet he stated that a display for 8 tens - 3 tens would also show 80 - 30.

When reading the symbolic problems, P would use one of two methods. For example, he read, "89 take away 53 equals 36," for

$$\begin{array}{r} 8\ 9 \\ -\ 5\ 3 \\ \hline 3\ 6 \end{array}$$

but for:

$$\begin{array}{r} 4\ 0 \\ +\ 2\ 3 \\ \hline 6\ 3 \end{array}$$

he read, "4 plus 2 is 6; 0 plus 3 is 3." This dual method also occurs when P solved mental computation problems. When asked to solve "23 plus 5," P said the answer was 28 because, "I added 5 and 3 and got 8. I already had 20." Yet he said "45 take away 32" was 12 because "I look away 3 from 4, equals 1 and, . . . 5 take away 2 equals 2."

ADDITION AND SUBTRACTION WITH REGROUPING

P quickly learned how to solve addition problems with regrouping with the sticks and the blocks. He was able to explain his manipulations and later, at the symbolic level, relate the numerals to the stick or block manipulations. P could show addition with regrouping on the abacus, but, at first, he was not very proficient at explaining his manipulations. Writing seemed to ease this and P began to choose the abacus over the sticks or blocks when he needed an aid to complete the symbolic worksheets. P seemed to solve the computation problems rather mechanically. For example when solving:

$$\begin{array}{r} 3\ 8 \\ +\ 5\ 7 \\ \hline \end{array}$$

P would talk to himself as he solved.

P: 8 plus 7 . . .8, 9, 10, 11, 12, 13, 14, 15.
(writes 15)

P: 3 plus 5 . . . 8.
(writes 80)

P: 5 plus 0 . . .5.
(writes 5)

P: 1 plus 8 . . . 9.
(writes 9)

$$\begin{array}{r} 3\ 8 \\ +\ 5\ 7 \\ \hline 15 \\ 80 \\ \hline 95 \end{array}$$

He could explain his work on the embodiments.

The shading on the addition with regrouping worksheets (see Appendix D) was very confusing for P. He usually ignored the shading rather than use it to determine the sum of the ones. However, the shading seemed to interfere with his counting. At one point, he said the shading meant "take away."

Before beginning subtraction with regrouping, P was completing worksheets involving both addition with regrouping and subtraction without regrouping. P carried the structure of the addition problems over to the subtraction problems. For example, P solved the following problem "the long way," as he called it.

$$\begin{array}{r} 58 \\ -\ 34 \\ \hline 4 \\ 20 \\ \hline 24 \end{array}$$

P's biggest problem when solving subtraction problems with regrouping with sticks or blocks at the enactive level was his forgetfulness. He would recognize the need to trade a bundle or long for 10 ones in order to solve the subtraction problem. He would add the 10 ones to the ones for the trade, but he would fail to remove the bundle or long. When asked to explain his work, P would say he had traded a bundle or a long, completely unaware of his error. P was never confident when using the abacus to show subtraction with regrouping.

Symbolically, P often did not indicate a change in the tens due to a trade, but he would remember it. For example, he solved:

$$\begin{array}{r} 57 \\ -\ 38 \end{array} \rightarrow \begin{array}{r} 5\not{7}^{17} \\ -\ 38 \end{array} \rightarrow \begin{array}{r} 5\not{7}^{17} \\ -\ 38 \\ \hline 9 \end{array} \qquad \begin{array}{r} 5\not{7}^{17} \\ -\ 38 \\ \hline 19 \end{array}$$

When completing worksheets involving both addition and subtraction problems, with and without regrouping, P often erred due to performing the wrong operation or due to arithemetic fact errors, but he usually recognized the need to "trade," as he called regrouping. Occasionally, he would fail to regroup and solve:

$$\begin{array}{r} 43 \\ -\ 26 \\ \hline 23 \end{array}$$

At the final interview, P was asked to solve addition and subtraction problems with regrouping involving three-digit numbers. He was not instructed how to do it, but was allowed to use the manipulative of his choice. P selected the abacus and correctly solved the addition problems, working the problems from left to right. For example, P solved:

$$\begin{array}{r} 156 \\ +\ 178 \end{array}$$

in the following manner:

1. Displayed 2 hundreds on the abacus.
2. Displayed 12 tens on the abacus.
3. Displayed 14 ones on the abacus.
4. Traded 10 tens for 1 hundred.
5. Traded 10 ones for 1 ten.
6. Stated the answer was 334.

Symbolically, P was not able to solve three-digit subtraction problems on the abacus. Without a manipulative aid, P could not solve three-digit addition and subtraction problems without regrouping.

XII. THE CASE OF R

by

PATRICIA CAMPBELL

BIOGRAPHICAL AND TESTING INFORMATION

R is a male child, born June 12, 1968. He is the fourth of five children with an older sister, age 18, two older brothers, age 11 and 12, and a younger sister, age 5. R lives with his mother, brothers and sisters in a federally supported housing project; he does not know his father. R's mother has a ninth grade education and works as a sales clerk in a department store.

R is a pleasant child, possessing a subtle sense of humor which is unnoticed by the other children. School is often a frustrating experience for R; he has very little self-confidence. R frequently copied the work of his classmates without any noticeable sense of guilt; however, if he was able to do his school work successfully on his own, he was very protective of it, not wanting the other children to copy his work.

R comes from disadvantaged circumstances and, when entering kindergarten he was already behind most of his classmates. He was unable to identify household items or animals and had no experience with shapes, colors, reading, or writing. During the first and second grade, R was in the Title I Program and had regular sessions with the school speech therapist. R seemed to have a perceptual problem as in first grade he wrote from right to left and by the end of first grade, he could recognize and name upper case letters but not the lower case alphabet. He also wrote the numerals 5, 6, 7 and 9 backwards. By the end of second grade, R could recognize and name both upper and lower case letters, but he still could not read. He continued to misspell his name frequently and write the numerals 5 and 6 backwards; however, he usually wrote from left to right. R was ranked lowest in the class by his second grade teacher. On the Comprehensive Test of Basic Skills (CTBS), administered at grade 2.6, R had a total reading scale grade equivalent of 1.7 and a language grade equivalent of .6. His mathematics scale was 1.9,

with a total overall CTBS grade equivalent of 1.5. R had an entering second grade Otis-Lennon IQ of 75. Tested at mid-year, R was classified as having "borderline learning disabilities"; the school psychologist recommended that he be retained in the second grade.

On the KeyMath test administered at the beginning of second grade, R had an overall grade equivalent of 1.6. He could not count items in a two-dimensional picture which utilized depth perception cues, however he could count accurately when the depth perception cues were not present. R correctly identified the missing numbers in the sequences 1, 2, ___, 4, 5 and ___, 6, 7, ___, 9, but he was able to tell the number which preceded 19 or identify the missing number in the sequence 98, 99, ___, 101. R was able to solve addition and subtraction problems of sums less than 10, but he did not attempt any computation problems involving two-digit numbers. R could not solve missing addend problems or oral computation problems. If accurate picture prompts were present, R could solve orally presented word problems.

The KeyMath test was again administered to R during the spring of the second grade. He received an overall grade equivalent of 1.9. At that time R knew the number which preceded 19, but he still could not identify the missing number in the sequence 98, 99, ____, 101, or count pictured items utilizing three-dimensional cues. R was able to solve addition and subtraction problems without regrouping, but did not know the basic facts for sums between 10 and 20. R could solve mental computation problems which involved addition or subtraction facts of sums less than 10 (e.g., 1 + 4 - 2) but he was unsuccessful on problems involving facts of sums greater than 10 (e.g., 5 + 3 + 6). R still could not solve missing addend problems or orally presented word problems without accurate picture prompts.

On the second grade PMDC test administered in the fall, R counted by touching objects. He could count from 6 to 15 and 35 to 46, but could not count back from 44 to 25. R could not count by tens. On the place value tasks, R always counted by ones, counting each bundle or chip as one. For example, when asked how many straws there were in 6 bundles of ten straws, he replied "6." R was unable to use counters to solve written addition or subtraction problems. R knew that 12 was more than 8, but he thought that 7 was less than 4, and that 19 was more than 31. When asked to order four numeral cards from smallest to largest, R placed them in the following manner: 5, 3, 2, 9.

In the spring, R again took the PMDC second grade test. R still counted by touching objects and could not count back from 46 to 35. However, he could count by tens from 10 to 100 and 26 to 96, but not beyond 100. R was able to perform the place value tasks involving bundles of straws and chips, counting by tens and ones. With the counters, R successfully solved the problems 2 + 4, 18 + 5 and 7 - 3; however, he added 2 + 3 + 7 to solve 23 - 7. R correctly ordered the elements of 2, 3, 5, 9 from smallest to largest and identified which number was more or less in the sets $\{7, 4\}$ and $\{8, 12\}$. However, R stated that 19 was more than 31.

As a second grader in the T1 teaching experiment, R was assigned to the multi-embodiment group (M) using sticks, Dienes blocks, and the abacus. R found work involving writing or reading to be very difficult, and he did not function well in the symbolic setting. However, he had no difficulty handling the concrete materials or pictures and usually was successful on tasks at the enactive or iconic levels. In the classroom, R rarely spoke; in the small M group, he volunteered his ideas, particularly if the task under discussion did not require reading. R would complete written worksheets assigned to him, but he was easily frustrated by corrections. If R had more than two items identified as errors on his worksheet, his frustration and tears precluded any additional assistance from the instructor. If R had only one or two items marked wrong, he was able to utilize suggestions from his instructor and correct his work. R rarely used the embodiments when completing symbolic worksheets because the other children in the group did not use them then. He needed them, however, in order to succeed with the tasks.

The following discussion of R's work in the T1 teaching experiment is organized under the following topics: place value, ordering, addition and subtraction without regrouping, addition and subtraction with regrouping.

PLACE VALUE

R had some initial difficulty counting from 10 to 15. Also when making displays with sticks or blocks to represent numbers with more than 5 tens or more than 5 ones, R would often lose his place in the counting sequence while finding sufficient sticks or units and would have to start counting over again. He gradually became proficient at making and counting displays, changing from one embodiment to another without hesitation, but preferring to use the abacus. At the enactive level, R became as efficient and accurate as the other children in group M on any of the embodiments. R had no trouble forming displays using sticks (blocks) which were prompted by displays or pictures depicting blocks (sticks). He initially had difficulty forming displays on the abacus from block or stick picture stimuli, but not from block or stick displays.

At the beginning of the T1 teaching experiment, R could not read and could not write any number greater than 11 with any consistency. Hence without picture prompts, he found filling in response sheets of the form:

_____ bundles and ____ sticks

____ tens and ____ ones

____ and ____ ones

to be very frustrating. Orally, R could describe a display in terms of the manipulative and then, if reminded the meaning of the words or prompted by pictures, he could fill in ____ bundles and ____sticks. The remaining statements were often confused:

20 tens and 3 ones

2 and 3 ones

R found no inconsistency in writing:

2 longs and 4 units

20 and 4 ones

42.

R initially had no understanding of the meaning of digits in written numerals representing numbers greater than 10. He often reversed the digits writing 02 for 20 or 61 for 16; he also confused sounds interchanging for example, 3 with 30, 13 with 30, 15 with 50 and 18 with 80, when writing or speaking. After learning the meaning of tens and ones and their representation in numerals, R was gradually able to read and write numerals using a translation process. From a display or picture, R could determine the number of tens and the number of ones and then record that information. Similarly, R could translate two-digit numerals into tens and ones, form a display with the embodiments, count, and then tell the number. Later he learned to translate without the embodiments. If asked to reach, for example, the numeral "35," R stated "3 and 5"; but if asked how much the number "35" was, he could translate "3 tens . . . 5 ones . . . 35." R never learned to write the numerals "12, 13, 15"; each time he wanted to write these numbers as solutions to his written exercises, R asked his instructor how to write them. Periodically throughout the year, R orally described displays or pictures as "___ ones and ___tens" rather than "___ tens and ___ ones."

In the fall R could not count by tens. He learned to translate from the number of tens depicted to the total number counting "1, 2, 3, 4, 5, 6, . . . 6 tens . . . 60," eventually becoming proficient at counting by tens. At first, counting by the tens led to occasional errors when counting by tens and ones as he would count "10, 20, 30, 40, 50" rather than "10, 20, 21, 22, 23." Continued experience with the three embodiments seemed to eliminate this counting error.

Larger numbers may not have had much meaning to R and the quantity of materials may have been distracting. He quickly learned to display quantities from 10 to 40 without recounting bundles or longs; however, he would sometimes stop and recount bundles or longs when making displays for numbers from 80 to 90. Later in the year with three-digit numbers, R usually grouped 10 longs in his displays to resemble a flat and quickly counted or formed displays with blocks. However, he was not as comfortable with sticks, and he occasionally recounted the number of bundles in his bundle of bundles. One day after breaking up 3 bundles of bundles and then counting the 30 bundles, R announced, "But that don't look like no 30." When counting, R usually hesitated when bridging hundreds.

R quickly learned to describe a three-digit numeral or a display (greater than 100) in terms of hundreds, tens, and ones. He also could adjust his displays so that they represented the same value but with "more tens" or "more ones." Although R was certain that 100 was 10 tens and conversely, that 10 tens was 100, R had no idea what number was, for example, 12 tens or 18 tens. After ordering pictures depicting 10 longs

(100), 11 longs (110), . . . 20 longs (200) and pictures depicting 1 bundle of bundles (100), 1 bundle of bundles and 1 bundle (110), 1 bundle of bundles and 2 bundles (120), . . . 2 bundles of bundles (200), R seemed to see a pattern for the first time. However, he still needed to use the embodiments in order to state how many tens were represented in a number between 100 and 200 or what number was represented by a given number of tens.

By translating in terms of the number of hundreds, tens, and ones, R could write or read three-digit numerals. Frequently, he used the abacus as an aid when writing these numerals from an oral stimulus.

Although R learned to read the words "hundreds, tens, and ones" and could use his manipulatives to display the values prompted by a written stimulus and record his display numerically, he could not function at a completely symbolic level unless the order of presentation was ___hundreds ___ tens ____ones. Without embodiments he was unable to successfully complete written exercises of the following form as he wrote:

17 tens = 17 hundred 7 tens

6 tens 1 hundred = 21 tens

10 hundred 19 tens = 190

Using Dienes blocks, R corrected his work on this exercise. Similarly, without embodiments he erred when completing exercises such as:

9 hundreds
2 tens
5 ones — 925

8 tens
2 ones
3 hundreds — 823

5 ones
0 tens
3 hundreds — 503

But when an abacus was present, he completed the exercise without error.

ORDERING

Ordering pictures depicting numbers from 10 to 20 involving differing manipulatives was a very difficult task for R. It was complicated by his inability to count from 10 to 15. Initially, abacus pictures were often misplaced or ignored in his ordering. Throughout the year, ordering pictures depicting larger numbers or ordering pictures representing nonconsecutive numbers was a very challenging task for R; if at all possible, he copied the ordering of the other children. R's method of ordering pictures representing consecutive numers was somewhat primitive as he would determine what number

should be portrayed in the next picture and then he would count every remaining picture until he found the one he needed. It was not until after Christmas that R learned to scan the pictures, noting the number of tens and ones, in order to facilitate his ordering and selection.

R could tell which of two displays or pictures was "more, less, fewer, or greater." He seemed to compare the numerousness of the sets rather than counting each set to determine how many. Shown a set in one embodiment, R could form a set that was more or less than the stimulus set. For example, when asked to make a set that was more than 27 (shown by blocks), R displayed 3 longs and 5 units (35). When asked to make a set that was less than the set of 13 sticks, R displayed 1 stick.

R never was comfortable using the formal terms "___ is greater than ___" or "___ is less than___." He would hesitate and forget the value of one of the displays or pictures before he completed the phrase. Also, if asked if a statement, "37 is greater than 41," was true or false, he would not know. But if he was asked which number was greater, 37 or 31, he could answer correctly.

At the symbolic level, R could not read the words "is" or "than"; hence, he had great difficulty placing the "is less than" or "is greater than" cards between numeral cards to describe displays. His method was to form the displays for each of the two numbers, write the numerals for each display, and then place either an "is less than" or an "is greater than" card. When told what the inequality card said, R would judge his work as correct or incorrect. If he felt it was incorrect, R would exchange the order of the displays and the numeral cards rather than change the inequality card.

Similarly R placed the inequality cards "$<$" or "$>$" as a memory task so that the symbol pointed to the smaller number, but he could not read the symbolic expression. If pressed to explain the symbolic statement, R would repeat the numbers over and over, and then make up an "is less than" or an "is greater than" statement. For example, R read: "16 $<$ 36" as "16, . . . 3 tens, 6 ones . . . 36, . . . 16 . . . 36 . . . is greater than 16." When writing the inequalities, R usually placed the signs vertically such as "56 $\vee$ 52."

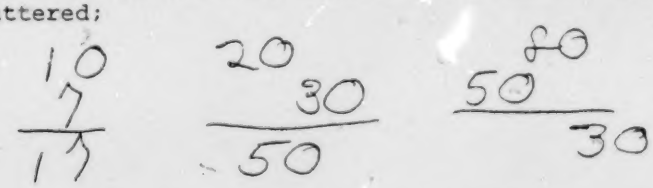
After showing a number with any of the manipulatives, if asked to show a number that was 1 more (less), 10 more (less), 2 more (less), 200 more (less), etc., R would adjust his display and count, correctly stating the adjusted number. He was not successful at these tasks at either a verbal or symbolic level without manipulatives; usually he would not attempt such tasks without a manipulative.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

R was able to use the embodiments to show both addition and subtraction problems without regrouping. He agreed that either horizontal or vertical displays with differing embodiments could be used to depict the same number sentence. The abacus pictures were confusing for R, but after some experience at using the abacus to show what the abacus picture depicted, he learned to interpret addition and subtraction conditions on the abacus pictures.

When adding without regrouping using either sticks or blocks, R would keep the tens and ones separated on the form board; he would then add the tens and ones simultaneously using both hands to sweep the tens together and the ones together. Symbolically, R usually added or subtracted the ones before adding or subtracting the tens. When ordering pictures to depict addition without regrouping (e.g., 21 + 13), R would first find the pictures to show each addend. Then he would count to find the sum (10, 20, 30, 31, 32, 33, 34), pointing to each item on the pictures as he counted. Finally, he would hunt for the picture that showed 34.

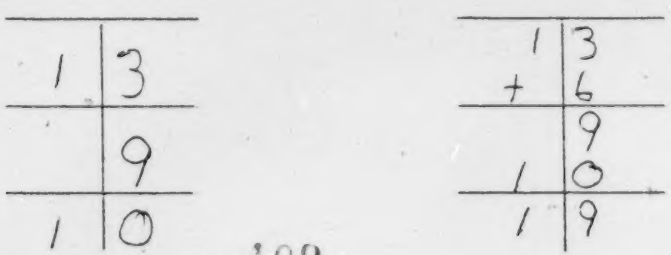
R had difficulty writing addition and subtraction problems without the structure provided by a place value chart. His numerals were somewhat scattered;



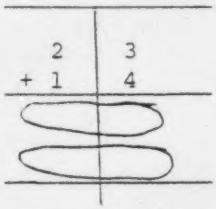
even when copying from the blackboard. He never placed either a + or a - sign.

An expanded form for the addition algorithm was initially confusing for R, and he did not know where to write his solutions.

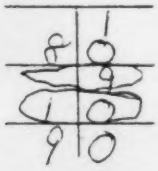
If R had to write and solve computation problems using the longer form on worksheets from pictured stimuli, he had difficulty writing the problem as well as noting his solution.



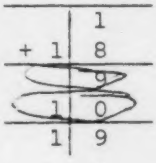
Ovals were then added to the structure of R's worksheets, and he was able to write both the problem and the solutions:



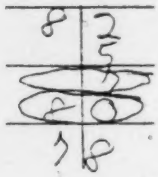
in the correct location. At this time R often reversed the order of the digits or wrote, for example, 60 for 16 or 90 for 19, while saying "16" or "19" aloud. For example, he wrote



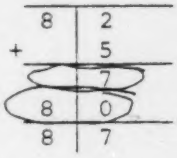
rather than



or



rather than



After completing several exercises of this type, R was able to solve addition problems without regrouping in the long form without pictures if the place value chart structure was present. If the structure was not present, he would not use the long form, rather he would solve for the total sum and write that solution.

Although R had difficulty writing the problems and reading them, he seemed to understand addition and subtraction without regrouping since he could explain the meaning of his written work.

If pictures were not present, R would solve addition and subtraction problems in a mechanical manner, talking aloud to himself. For example, when solving:

$$\begin{array}{r} 80 \\ -\ 40 \\ \hline 40 \end{array} \qquad \begin{array}{r} 52 \\ +\ 37 \\ \hline 89 \end{array}$$

R stated: "0 take away . . .0 (writes 0). 8 take away 4 . . . 4 (writes 4) . . . 2 plus 7 . . . 9 (writes 9). 5 plus 3 . . . 8 (writes 8." If R did not know the solution to the basic fact contained in the problem, which he usually did not, he used his fingers or made tally marks to find the solution.

During oral drill, R seemed to understand the relationship between the basic facts for addition and subtraction such as "8 take away 3" and "8 tens take away 3 tens." However, he could not transfer these solutions to oral problems such as "80 take away 30." But R stated that the display for 8 tens - 3 tens would also show 80 - 30.

R did not know the basic facts. For problems involving facts with sum less than 10, he would use his fingers. For addition problems of sum from 10 to 20, R would solve by tallying and, for subtraction facts, he would "subtract up."

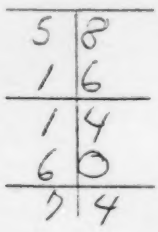
$$\begin{array}{r} 16 \\ -\ 7 \\ \hline 11 \end{array} \qquad \begin{array}{r} 15 \\ -\ 9 \\ \hline 14 \end{array} \qquad \begin{array}{r} 12 \\ -\ 5 \\ \hline 13 \end{array}$$

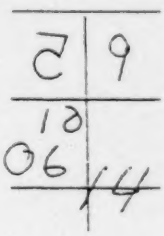
After instruction concerning counting on a method of adding and subtracting, R was able to solve the basic facts from sum 10 to 20. However, in the spring of the year, R began trying to subtract by counting back and became very confused. He reverted back to "subtracting up." R was never able to solve these mental computation problems which he could not solve on his fingers. On written exercises containing both addition and subtraction problems without regrouping, R would sometimes err by performing the wrong operation.

ADDITION AND SUBTRACTION WITH REGROUPING

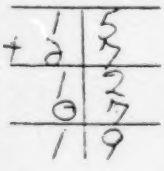
R quickly learned to do addition problems with regrouping using the sticks and the blocks. He initially needed step-by-step directions in order to use the abacus for addition problems with regrouping and was frustrated when using that manipulative. After practice with all three embodiments, R seemed to see the similarity between his trading activities with the sticks or blocks and the trading on the abacus and would choose the abacus over the sticks or blocks at the enactive level.

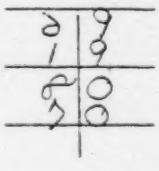
R seemed to have more difficulty writing addition problems or writing the solution, than he had determining the solution. For example, R's initial attempt at writing

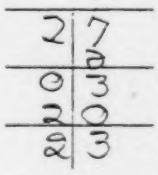




Gradually, with further use of the blocks and sticks on the form board, recording his manipulations, and further instruction with pictures and writing, R learned to solve and write the solutions to addition problems with regrouping, if a structure was present. Periodically he would err by reversing digits, writing 30 for 13, 80 for 18, or a combination of these errors.





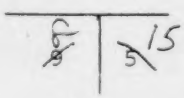


If the place value chart structure was not present, R added by columns:

$$\begin{array}{r} 2\ 3 \\ +\ 5\ 8 \\ \hline 7\ 11 \end{array} \qquad \begin{array}{r} 7\ 9 \\ +\ \ 9 \\ \hline 7\ 18 \end{array}$$

Subtraction problems with regrouping at the enactive level were more difficult for R than the addition problems as he would sometimes forget to remove a ten, after "trading" for 10 ones or he would lose his train of thought, forgetting that he was to subtract, when counting 10 ones for a trade. Again the abacus was more challenging than either the sticks or blocks, and he initially required step-by-step guidance on its use. With practice, R became proficient solving subtraction problems with regrouping on the abacus.

At the symbolic level, R never understood subtraction problems with regrouping. He was able to make a display to represent a number, trade a ten for 10 ones, and then explain why his new display still showed the same number, recording his work as:



But R never was able to apply this in a symbolic subtraction situation. When solving written subtraction problems with regrouping, R refused to use the manipulatives which he needed, since the other children in the group were not using them. Instead he would solve:

$$\begin{array}{r} 4\ \overset{1}{5}\overset{15}{} \\ -\ 1\ 9 \\ \hline 2 \end{array}$$

and then ask, "Is this right?" After being reminded that he had to trade something in order to have 15 ones, R would complete the problem. However, in testing situations without an embodiment, R simply "subtracted up."

$$\begin{array}{r} 3\ 5 \\ -\ 1\ 8 \\ \hline 23 \end{array} \qquad \begin{array}{r} 8\ 6 \\ -\ \ 9 \\ \hline 83 \end{array} \qquad \begin{array}{r} 4\ 0 \\ -\ 1\ 6 \\ \hline 32 \end{array}$$

R's lack of understanding of regrouping may be best illustrated by his attempts to solve regrouping problems involving three-digit numbers. He could not solve the problems on the abacus and symbolically he solved:

$$\begin{array}{r} 4\ 2\ 6 \\ +\ 1\ 8\ 2 \\ \hline 5108 \end{array}$$

saying "4 and 1 makes 5; 2 and 8 makes 10; 6 and 2 makes 8." Similarly he solved:

$$\begin{array}{r} 2\ 3\ 7 \\ -\ 1\ 7\ 6 \\ \hline 141 \end{array}$$

saying, "2 take away 1 is 1 and 7 take away 3 equals 4 . . . And 7 take away 6 equals 1."

XIII. THE CASE STUDY OF S

BY

CYNTHIA CLARKE

BIOGRAPHICAL AND TESTING INFORMATION

S, a female, was born May 25, 1968 and has a sister four years younger. Her mother is completing her undergraduate degree and works parttime at a department store. Her father, who is physically handicapped, attended law school and works in another city. Perhaps because of her father's handicap, S is very sensitive and extremely kind to those who are less fortunate than she is. She would frequently seek out the child on the playground who had no one to play with.

S is above average in her class, although her Otis-Lennon IQ of 96 reflects average ability. Her parents strongly encourage her to succeed in school. S is unhappy being the smartest in a group and is embarassed by being singled out for her academic achievements. She would often encourage the teacher to find errors in her written work and would write the total number wrong at the top of a worksheet page if the teacher did not.

A sociogram of S's second grade class showed her to be extremely popular especially with the girls. She did not like to get dirty and played mostly "girl" games, such as school and jacks. S possesses a strong imagination and a charming sense of humor. She exhibited evidence of religious influence, often writing stories about God and Jesus and indicating that she read the Bible at night.

S is anxious to please the teacher and would prefer not to attempt a new type of problem rather than to chance doing it incorrectly. On achievement tests, the regular classroom teacher often found it necessary to remind S that if she could not do a particular problem she should go on and try the next one.

In the first grade, S was one of a few first graders placed in a second class due to space limitations. During the second grade she showed marked improvements in all subject areas according to her CTBS (Comprehensive Test of Basic Skills) scores. Her mathematics subscale score increased from a grade equivalent of .8 at the end of the first grade to 2.8 at the end of the second grade. Increases in the other areas were of similar magnitude (from 1.9 to 3.9 in reading, 2.3 to 4.1 in language, 1.7 to 3.4 for total scale).

The KeyMath Diagnostic Arithmetic Test, administered at grade 2.0, yielded a grade equivalent of 2.2. S scored above second grade level in numeration, orally filling in the blanks for ___, 6, 7, ___, 19; and 98, 99, ___, 101. She could perform simple addition, subtraction, and word problems with pictures and symbolic addition, subtraction, and missing addend problems (sums less than 10). She correctly answered two of the difficult numerical reasoning problems, but could not handle mental computation involving more than two numbers.

On the PMDC Grade 2 Mathematics Test, also administered in the fall, S was able to successfully count picture sets, count from 6 to 15 and 35 to 46, count back from 6 to 1, and from 44 to 33. She could count by tens to 100, but starting at 26 she could only count by tens to 46. S could tell and write the number of sets of straws and color-coded chips by counting by tens and ones and constructed a set of straws of a given number. She could not, however, construct a set of straws or chips corresponding to a written numeral or construct a set of a given number of chips in response to oral instructions. S solved simple addition and subtraction problems using beans, but miscounted on the more difficult two-digit plus one-digit addition problems. She added on the two-digit minus one-digit subtraction problem and again miscounted. She could order 2, 3, 5, 9, tell which of 8 and 12 was more and tell which of 7 and 4 was less. She indicated, however, that 19 was more than 31.

The administration of the KeyMath test at grade level 2.8 yielded a grade equivalent of 2.6. S showed a slight improvement in numeration, answering one additional item correctly. She demonstrated ability to do addition and subtraction with and without regrouping. She could now do mental computation involving three addends, but not where both addition and subtraction were involved. Although she had done so in the fall, S was unable to solve any of the difficult numerical reasoning problems.

The spring administration of the PMDC Grade 2 Mathematics Test showed S able to perform every task correctly, except that she only counted from 26 to 106 by tens (counting to 126 was corrected) and she still chose 19 as being more than 31.

As a second grade student in the T1 teaching experiment, S was assigned to the group 2M using six manipulatives: popsicle sticks, unifix cubes, Dienes blocks, grid paper, abacus, and colored chips. S was the highest ability child of the four who remained after Christmas and was always attentive even though often not challenged. She was always first to complete symbolic exercises and made sure that her work was neat and orderly.

The remainder of this report will be organized under four topics: place value and numeration, ordering, addition and subtraction without regrouping, and addition and subtraction with regrouping.

PLACE VALUE AND NUMERATION

S could count to 50 at the beginning of school and by Thanksgiving had demonstrated a firm understanding of tens and ones. This understanding is exemplified by the fact that she could rapidly display any two-digit number with any of seven aids, including one with which she was not familiar. Changing from one manipulative aid to another caused her no difficulty. S could describe any number less than 100 in terms of tens and ones:

$$\underline{4}\ \text{tens} + \underline{6}\ \text{ones}$$

$$\underline{40} + \underline{6}$$

$$\underline{46}$$

When asked why she wrote 40, she replied "Because there are two ways for 40: 4 tens and 40." She could give these three equivalent expressions for any number orally, as well.

S knew that

tens	ones
3	15

was 45 without counting, and could trade with the manipulative aids to show it. When asked how she knew she said "There is a ten in 15 ones." She was also able to show

tens	ones
4	2

as 3 tens and 12 ones, and knew that there were still 42.

At the beginning of the unit on three-digit numeration, S could not count past 190 by tens. She picked this up rapidly and had only slight difficulty with exercises like the following:

170

__1__ hundred __7__ tens

__17__ tens

She had difficulty when the order of the words or statements was altered, however, and was never comfortable with exercises such as:

19 tens = ____ tens, ____ hundreds

or

____hundreds, ____ tens = 20 tens

After a lesson on counting by 10's and 100's to 1000, S was unable to show 300 with bundles of counting sticks. She put out 13 bundles, counted to 130 then after much hesitation and prompting she realized she needed more. She then grouped the original 13 bundles into sets of 10 and 3 and proceeded to complete the task correctly by displaying 3 sets of 10 bundles. When asked to show 22 tens with Dienes blocks, S did not know what to do and required help. After each student showed 22 longs, we counted to 220 as a group. When the question "How many tens are there in 220?" was asked, S figured out the answer after a slight hesitation, while the others had to recount the longs.

By the end of the year, S could write the number (0-1000) displayed in a picture, display a given number with any of the six manipulatives and circle the pictured objects that correspond to a particular digit of a 3-digit numeral.

At the end of the year, S was asked to read and show with sticks the following:

hundreds	tens	ones
2	13	4

She read it correctly, but her display looked like this:

She interpreted the thirteen tens as one ten and three ones. After some help she displayed thirteen tens. When asked how much altogether, she traded ten of the bundles for a hundred and said, "334."

S read and displayed

2	4	5

correctly with Dienes blocks. She did not understand what was meant by "Show 245 in another way," but responded correctly to "Make more tens" by trading a hundred for 10 tens, and said "There are still 245."

S could not tell how many tens altogether in 150 when shown the numeral. After showing 150 with one flat and five longs she answered 15. She rejected the equality

245 = 1 hundred, 13 tens, 15 ones .

but did not know why it was not true. This tendency to make snap judgements when presented with the unfamiliar was typical of her behavior in response to new situations. When asked if a picture of 14 bundles and 16 sticks showed 156, she answered "No, it's not right because there are no hundreds and more than 6 ones."

While often unable to view numbers greater than 100 in more than one way as shown above, S seems to have a good understanding of place value. Her firm grasp of 2-digit numeration proved extremely helpful to her as she tackled order, addition and subtraction problems.

ORDERING

S performed well on all ordering of picture tasks. She usually started wth the smallest number and looked for each one in order often remembering the number of some of the pictures without having to recount. When ordering a set of stick displays for 7, 23, 34, 43, 75, and 90, she put the picture for 7 just before that for 75, evidently interpreting the 7 as 70. S was the only one of the group who correctly ordered 13, 14, 20, 21, 50, 51, 52, 99.

S knew the meaning of the phrases greater than, more than, and less than at the beginning of the year and could manipulate the words (orally and symbolically) and the symbols $<$ and $>$ to make true statements. After she developed an understanding of two-digit numerals she could compare any two numbers and make oral or symbolic "less than" or "greater than" statements about them. When asked "Which is more, 52 or 56?" S responded that 56 was more "Because 2 comes before 6 and as the numbers go by they get higher."

At first, S needed the objects when working exercises requiring her to start at a given number and write the numbers which follow consecutively by ones, tens, or hundreds, or even to perform such a task orally. She soon began noticing the patterns, observing which digit changes, and was able to abandon the manipulatives.

After the unit on two-digit numeration, S was unable to write the number that is ten more, one more, or one ten more than a given number. She did not understand what was being asked, and simply copied the given numeral. She

showed 27 with 2 cups of ten pieces of candy each and 7 single pieces of candy, but could not show one more. She did, however, show ten more correctly after one more than 27 was shown to her.

By the end of the year S could write, tell, or show with any of 7 manipulative aids the number which is one more (less), ten more (less), or one hundred more (less) than a given number (less than 900, written or given orally).

ADDITION AND SUBTRACTION WITHOUT REGROUPING

S could do simple addition and subtraction with sums of ten or less on her fingers or with manipulatives at the beginning of the year, but had memorized only a few addition facts and no subtraction facts. When presented with one of the more difficult addition facts (sum 10), she insisted she could not do it because she did not have enough fingers. After instruction on using her fingers to count on for addition and count back for subtraction, S was able to compute any addition or subtraction fact rapidly. She was so fast and accurate that she took a long time committing them to memory.

S had no difficulty with symbolic addition involving one or two digits, without regrouping. She could demonstrate using any of the aids, and could usually explain how she did a problem in terms of tens and ones. However, when asked "How did you get the answer?" after working the problem:

$$\begin{array}{r} 40 \\ +\ 23 \\ \hline 63 \end{array}$$

she responded "I put together 40 and 23." When matching pictures with addition problems, S would compute the answer first, then look for the picture. S could work two-digit addition and subtraction problems mentally, but often could not explain how she got her answer. She could, however, demonstrate with objects. She was able to write addition problems from pictures or actual displays, but did not use the pictures or objects to solve the problem, just to determine the addends.

After working a problem like the one below, S would read the problems as:

3 tens + 2 tens = 5 tens 1 one + 3 ones = 4 ones.

$$\begin{array}{r|r} 3 & 1 \\ +\ 2 & 3 \\ \hline & 4 \\ 5 & 0 \\ \hline 5 & 4 \end{array}$$

However when the vertical line was removed she would read the problem as "31 + 23 = 54."

S continued to put in the middle step in the addition algorithm for a long time, even after she learned to regroup and became aware of the necessity of the middle step in problems requiring regrouping. This was typical of her behavior. She did what she was told, rarely asked questions, and would not change techniques unless instructed to do so. Once, when shown a picture showing 25 + 2, and the problem

$$\begin{array}{r} 25 \\ +\ 2 \\ \hline 45 \end{array}$$

S said the problem did not go with the picture, but could not change it to make it correct.

S usually had no trouble writing a subtraction problem from a picture or display. As with addition, she only used the picture or display to determine the subtrahend and minuend, not to solve the problem. Interpreting pictures depicting subtraction problems with zero differences caused S difficulty, especially when they were shown on the abacus.

S could solve any two-digit subtraction problem without regrouping and could demonstrate the solution with any of the six manipulatives or one with which she was not familiar.

ADDITION AND SUBTRACTION WITH REGROUPING

Before instruction on regrouping, S was asked to do an addition problem requiring regrouping. She made the common error as shown below:

$$\begin{array}{r} 39 \\ +\ 23 \\ \hline 512 \end{array}$$

She did, however, express concern that her answer might not be correct. She attempted to work the problem on the abacus, but could not tell how much 5 tens and 12 ones were altogether.

When asked to work a subtraction problem requiring regrouping before instruction, S again gave the typical incorrect response:

$$\begin{array}{r} 43 \\ -\ 24 \\ \hline 1 \end{array}$$

She was concerned about "3 take away 4," however, and could not arrive at an answer with the Dienes blocks.

After a short period of instruction, S could work 2-digit addition problems with or without regrouping and could demonstrate them with any of the manipulatives. S often had difficulty with situations requiring her to reflect on the correctness of a problem completed by someone else. She always

knew if a problem was worked incorrectly but rarely could change it to make it correct. For example, when presented with the problem below, the discussion went as follows:

$$\begin{array}{r} 34 \\ +\ 29 \\ \hline 513 \end{array}$$

Q: Read this for me.

S: Thirty-four plus twenty-nine equals fifty-thirteen,

Q: Is the problem worked correctly?

S: No . . . I can't think of this number . . . fifty-thirteen.

Q: What would you do to make it right?

S: Add them together . . . it equals 63.

S could work a problem of this type correctly, but could not make the necessary changes to one worked incorrectly.

Before the regrouping algorithm for subtraction was introduced, the students practiced trading one ten for ten ones and vice versa and recording the results. At first this was done at the manipulative-oral level. S had some difficulty verbalizing the equality before and after trading, especially when trading one ten for ten ones. After instruction on recording the trading, the students were given a worksheet requiring them to show trading one ten for ten ones for each number. S did three problems correctly, then did the following:

8	4
~~0~~	~~5~~

When questioned, she saw her error and corrected it.

After a considerable amount of practice, subtraction with regrouping was easily mastered by S. She had no difficulty working a problem with any of the manipulatives and explaining the regrouping procedure. She sometimes traded when it was not necessary, even on problems such as 15 - 8, which was consistent with her rigid use of other strategies she was taught. After spending a great deal of time on subtraction, S looked at

$$\begin{array}{r} 2\ 9 \\ +\ 5\ 3 \\ \hline \end{array}$$

and said "We have to trade for tens this time." She also traded in an addition problem, but corrected her error. These again, are signs of inflexibility to change problem situations or strategies.

When asked to work three-digit addition and subtraction problems requiring regrouping with manipulatives (without instruction), S usually realized she needed to trade but often did not, or traded incorrectly.

S worked

$$\begin{array}{r} 345 \\ +173 \\ \hline \end{array}$$

with Dienes blocks.

She did it correctly, but did not trade, although she said she needed to. When she counted she got 428 as her answer.

S used beans (single, bags of 10, and bags of 100) to solve

$$\begin{array}{r} 324 \\ +195 \\ \hline \end{array}$$

After combining addends, she said the answer was "Four hundred eleven . . . I don't know this number." Finally she put ten 10's with the hundreds, and said the answer was 519.

S had greater success when not using the objects. She began reading a three-digit numeration problem as "6 + 2 is 8," but when asked to "Read the whole problem," she did so correctly. She used the shortcut to the addition algorithm which was not taught in her math group.

Subtraction caused S greater difficulty. She traded correctly when working:

$$\begin{array}{r} 524 \\ -\ 183 \\ \hline \end{array}$$

but subtracted incorrectly obtaining an answer of 451. She knew trading was necessary to work:

$$\begin{array}{r} 245 \\ -\ 148 \\ \hline \end{array}$$

but worked from left to right, so she did not have anything to trade.

After it was suggested that she work right to left, she traded correctly but got confused and never reached an answer.

S worked

$$\begin{array}{r} 436 \\ -\ 175 \\ \hline \end{array}$$

correctly with a manipulative aid consisting of bean bags and beans, but needed help deciding what to trade.

When given a problem to work without manipulatives, S reverted to the "take the smaller number from the larger" strategy which she had used on two-digit regrouping problems before learning the algorithm:

$$\begin{array}{r} 237 \\ -\ 176 \\ \hline 141. \end{array}$$

While S is a bright girl who is learning very young how to please the teacher (i.e., do what you are told, no matter what), her inability to reason through unfamiliar problems and to alter her strategies in new situations will probably hinder her mathematical progress in the future years.

XIV. THE CASE OF T

BY

CYNTHIA CLARKE

BIOGRAPHICAL AND TESTING INFORMATION

T is a male child and was born October 4, 1968. He has two brothers, ages 21 and 22, and two sisters, ages 2 and 14. He lives with his mother, who receives welfare, in a housing project, but also stays with his aunt occasionally.

T is an easy-going boy, is easily distracted, and has great difficulty staying on tasks. He appears hungry for attention and tried to earn it by misbehaving. His inability to finish anything prompted his second grade teacher to have him tested for learning disabilities. The results showed him to be of low ability with no learning handicaps.

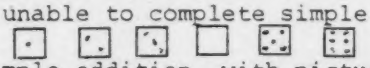
T had problems with drooling and being slightly overweight throughout the first grade and the beginning of the second grade. There was improvement in both of these areas during the second grade although he constantly had something in his mouth, usually paper, rubber bands, or paper clips.

T avoided joining groups, preferring to play on the outskirts but was very friendly. He chose one boy and one girl as friends for the class sociogram and was chosen by two children as a friend. He enjoyed building

things with blocks, playing with cars, and listening to records. He watched a great deal of television at home and made frequent references to his favorate shows.

T's Otis-Lennon IQ is 88. His CTBS (Comprehensive Test of Basic Skills) test scores reflect below-average competencies. At the end of the second grade, his reading grade equivalent was 1.2 (3rd percentile). He was still reading the same book that he was reading at the end of the first grade. His language skills also showed little improvement with a grade equivalent of .7. His mathematics and total scores increased from .1 to 1.8 and .1 to 1.4 respectively from the end of the first to the end of the second grade.

On the PMDC Grade 2 Mathematics Test, administered in the fall of the second grade, T was able to count 3 horses and 7 cows, but miscounted 13 dots (14) and 10 animals (9). He could not rote count past 40, or count back from 44 past 40, but could count back from 6 to 1. He counted to 80 by tens, but could not count from 26 by tens, even with prompting. He showed no knowledge of place value or two-digit numeration and could not determine, or write the number for sets of straws or chips, or construct sets of a given numer with either. He could not use beans to show even simple addition and subtraction, but knew 2 + 4 = 6 without beans. He knew 12 was more than 8 but chose 19 as more than 31. He ordered 2, 3, 5, 9 correctly but did not understand the meaning of less than.

The KeyMath Diagnostic Arithmetic Test, also administered in the fall, yielded a grade equivalent of 1.0. T showed little understanding of numeration. He was unable to complete simple patterns such as 0 x 0 x 0 ___ or 
He could do simple addition, with pictures and symbolically, but could subtract only with pictures. He answered 1 + 1 (given verbally) correctly but missed 2 + 2 and was unable to do numerical reasoning problems such as 3 + 2 = □ and 1 + □ = 2. T was able to do only one word problem, even when accompanied with pictures.

In the spring, the PMDC Grade 2 Mathematics Test was administered again. T again miscounted 13 dots and 10 animals as well as 7 cows. His rote counting skills improved; he could now count by tens to 130 and from 26 to 96. He still could only count back from 44 to 40.

He demonstrated understanding of place value, correctly responding to all the color-coded chips and straw items, counting by tens and ones. He used beans to correctly answer simple addition and subtraction problems, but would not use beans to solve 18 + 5, and answered incorrectly. He made 2 sets of 23 and 7 for 23 - 7 and got an answer of 7. He showed understanding of less, but still chose 19 as being more than 31.

The KeyMath test administered at grade level 2.8 yielded a grade equivalent of 2.5, an increase from the fall of 1.5. T showed considerable improvement in numeration skills, orally filling in ___6, 7, ___9; ___19; and 98, 99, ___, 101. He showed knowledge of ordinal numbers through five. He demonstrated ability to do two-digit addition with regrouping. He could handle basic subtraction facts (14 - 6), and two-digit subtraction without regrouping (76 - 12),

but could not do a regrouping problem (25 - 16 = 19). T was now able to do mental computation involving three numbers and both addition and subtraction if the numbers were small. His numerical reasoning ability greatly improved, as evidenced by his answering three of the difficult problems correctly (such as 2 - △ = 1, 3 - △ = □). He answered four word problems with pictures correctly.

As a second grade student in the T1 teaching experiment T was assigned to the grouping using six manipulatives: popsicle sticks, unifix cubes, Dienes blocks, grid paper, abacus, and colored chips. At the enactive level one could not distinguish T from the brightest child in the group. He operated efficiently and rapidly with the manipulatives. Pictures caused him some difficulty but the symbolic work was frequently his downfall. He was very slow, and often needed to use the objects to work symbolic problems when the other children did not. Unfortunately, T did not like to be the only one using the manipulatives and felt it was a weakness to use them. This caused problems because he would abandon them too soon. T wrote his 2's, 4's, 6's and 9's backwards at the beginning of the year and still frequently (though not consistently) write his 4's and 6's backwards at the end of the year. Writing numerals was quite a struggle for T and slowed him down a great deal.

The remainder of this report will be divided into the following four sections: place value and numeration, ordering, addition and subtraction without regrouping, and addition and subtraction with regrouping.

PLACE VALUE AND NUMERATION

T began the second grade with a very poor number sense. He could not determine the number of a set of as few as four objects without counting. He had no problem believing that you could start with five things, take two away and still have five left if that was the answer he got using the manipulatives incorrectly. T was at a disadvantage when working exercises such as:

2 longs + 3 units

___ + ___

___

because he was a very poor reader. Later on in the year, exercises like those below continued to cause him difficulty, again due primarily to his reading problem.

19 tens = ___ tens, ___ hundred

6 tens, 1 hundred = ___ tens

After the unit on two-digit numeration T was able to read, write, or display with any of the six manipulatives any number less than 100. He was also able to transfer this knowledge to a manipulative with which he was not familiar: covered cups of candy. His understanding of tens and ones came on rather rapidly and brought with it an improved number sense. For example, once when asked to show 17 with sticks he put out a bundle of ten and nine single sticks. After counting to 19 he realized his error and took two of the sticks away without recounting.

T was able to complete exercises such as:

4 tens and 6 ones

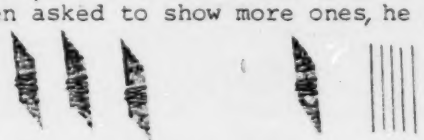
$$\underline{40} + \underline{6}$$

$$\underline{46}$$

T could not, however, explain why he put 40 where he did. This was typical of his inability to verbalize his reasons behind symbolic responses. As mentioned previously, T excelled with the manipulatives but found the symbolic forms a constant struggle.

T caught on rapidly to number patterns, such as 2 + 5 = 7, so 20 + 50 = 70. When the transfer was from one to ten, the oral pattern was not as obvious and caused him difficulty. He relied heavily on the way the numbers sound. After the group worked the problem 30 + 60 in vertical form, T anticipated that the answer to 30 + 6 was 90 and was surprised when he counted and found that his guess was incorrect.

When asked to read $\overline{3 \mid 15}$ T said "3 tens plus 15 equals 45." His display is shown below. When asked to show more ones, he put back a bundle and put out 10 single sticks.



After showing $\overline{4 \mid 2}$ he was asked to show it in a different way. He kept one bundle and counted out the rest in ones, but said there were still 42.

At the beginning of the unit on three-digit numeration, T had difficulty counting by tens past 190. He repeatedly said 180, 190, 120. He also had trouble observing the connection between 13 and 130 and insisted for a long time that there were 113 tens in 130.

T required constant help on two worksheets to complete exercises like the one below when one of the lines was given:

___ tens

___hundred, ___ tens

_____

On a worksheet requiring him to rename multiples of ten, given pictures, T wrote:

100 1 hundred, 0 tens

110 1 hundred, 1 ten

120 2 hundred, 0 tens

130 3 hundred, 0 tens

140 4 hundred, 0 tens

He was unable to follow the pattern, so he made up his own. Other frequent mistakes were:

130

1 hundred, 13 tens

30 tens

and

130

1 hundred, 30 tens

30 tens

T did not respond when asked to "Show 22 tens with blocks," even after a long was held up and the question "How much is this?" was asked. Finally he copied the others. After the group counted to 220, the question "How many tens in 220?" was asked. T could not remember what the initial task was and had to recount the longs.

At the end of the year, T could read, write or display any number less than 1000. He was able to determine and write the number of a display and circle the digit corresponding to the circled objects in a display. He could also write the number corresponding to:

8 tens

2 ones

3 hundreds

T read $\overline{2\,|\,13\,|\,4}$ correctly as 2 hundreds, 13 tens, and 4 ones, but when showing it with blocks, he interpreted the 13 tens as one ten and three ones. Finally with help, he was successful. When asked "Can you do something to make more hundreds?" T added a flat. After taking the flat back, the interviewer asked "Can you trade to make more hundreds?" and T traded correctly saying there were 334 altogether.

T read $\overline{2\,|\,4\,|\,5}$ correctly, traded to make more ones and said there were still 245. T knew there were 15 tens in 150 (when shown the numeral) but could not explain how he knew.

ORDERING

T began the year thinking greater and less meant the same thing. Once he got this straightened out he usually had no difficulty pointing to the set which had more, less, or the greater number. Verbalizing these situation was difficult for him. He would make statements such as "31 is greater than 25. 25 is greater than 31." The words just weren't meaningful to him. He once said "49 is gooder than 35."

Once when a set of 12 chips was displayed, each child was to make a set which was greater than the given set. T showed 148 with tens and ones. When he was asked "Is there a smaller number that is still greater than this set?" T said "There ain't nothing greater than this (pointing to his set)."

Eventually T acquired the skills necessary to compare any two numbers by looking at the displays and could make verbal "greater than" and "less than" statements about them. However, he was clearly not ready for the symbolic treatment of these concepts. The phrases "greater than" and "less than" caused him a great deal of trouble, and the symbols $>$ and $<$ were even harder for him. He frequently could construct the verbal statement correctly but could not use the symbols.

T was not very successful at ordering picture sets, primarily because he did not have a technique for organizing his results. He would count all the pictures and could compare any two of them, but often did not know where to start and refused to separate the pictures he had put in order from those that remained. After receiving help to get started he could usually complete the task but was extremely slow. His difficulty seems to be operational rather than conceptual.

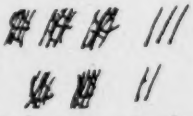
When ordering a set of stick pictures for 7, 23, 34, 43, 75, 90, T put the 7 just before 75, evidently interpreting it as 70. He had a tendency to concentrate on the numerousness of the objects. For example, one ten and nine units appeared to be more than 3 tens. This was helpful when ordering decades, such as 40 - 50, but a hazard when ordering 13, 14, 20, 21, 50, 51, 52, 99. T put 20 and 21 first. He also had trouble deciding where to put the multiples of ten; in this case he put 50 after 51 and 52.

During the unit on two-digit numeration, T was asked to show 50 with the grid paper. He rapidly shaded in 5 rows. The when asked to show 52 he shaded in 2 more squares and said "Two more is 52" without counting. However, at the end of the unit he was asked to write the number which is one more (less) or ten more (less) than a given number. He did not understand what he was supposed to do and merely copied the numerals. He did, however, show 27 with 2 cups of 10 pieces of candy each and 7 single pieces, and then showed one more, saying there were now 28. Similarly he showed ten more by adding a cup of ten and knew there were then 38.

By the end of the year, T could show 1, 10 or 100 more (less) than any given number. He had trouble, however, bridging hundreds, especially on "less" tasks such as 10 less than 309. T was first in the group to catch on to the oral patterns involved in starting at a certain number and counting forward to backward by 1's, 10's, or 100's. At the beginning he needed to use objects, but eventually he could do symbolic exercises without them, only occasionally using the abacus for exercises involving bridging of hundreds.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

T understood addition at the enactive level long before he was ready for the symbolic algorithm. He could correctly answer orally presented problems such as 30 + 40, but needed objects when presented with the same problem in the symbolic mode. Even when pictures were given, T initially found it necessary to duplicate the picture with the manipulatives. He could not combine the sets in his mind, but had to do it physically. For example, when counting the two sets below, he said "10, 20, 30, 31, 32, 33, 20, 30, 31, 32."



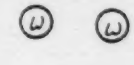
He even had difficulty after it was suggested that he count the tens first.

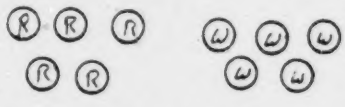
T saw no reason for the middle step (see the problem below) in the algorithm when regrouping was not involved. When adding symbolically he did not use the middle step at all to determine the answer, but used manipulatives or a picture if one was given. Hence he saw nothing wrong with:

$$\begin{array}{r} 12 \\ +\ 23 \\ \hline 5 \\ \ 35 \\ \hline 35 \end{array}$$

After completing such a problem, T frequently could not answer the question "What is 24 + 13?"

Eventually T could add symbolically and demonstrate what he had done with objects. Problems involving zeroes caused him some difficulty at the symbolic level, although he could handle them orally. After much 2-digit plus 2-digit addition practice T had difficulty with 1-digit plus 2-digit problems. When asked to tell the problem shown by:



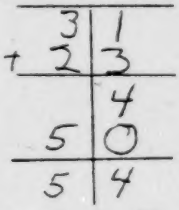


he could not do it. The single-digit addend, especially on top, was new and unfamiliar to him.

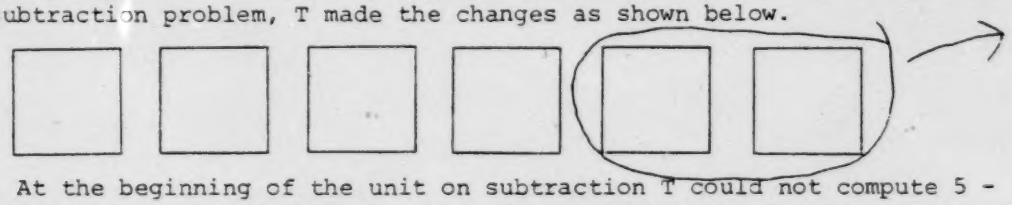
T did not connect the short form with the long form (with middle steps for sums of tens and ones) for quite a while and worked them as two separate problems even though they were side by side.

Verbalizing what he had symbolically was difficult for T. His reason for doing something was often "I know." T could not handle mental problems, even two-digit plus one-digit, but usually "guessed" an answer that was within three of the correct answer. When working addition and subtraction problems using bags of 10 beans each and single beans, T used single beans for both tens and ones and arrived at the correct answers. This was unusual since T did not usually interpret a number as separate digits, but as the sum of tens and ones.

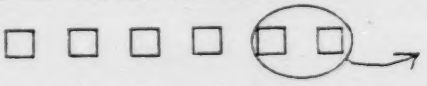
T could construct addition problems from pictures, such as the one below which he wrote from a Dienes block picture.



He read the problem as "30 + 1 is 30, so 20 + 3 is 23, so now add the ones together, see, it's 4. So add the tens together, so it's 50. It's 54." He also exhibited the ability to change a completed addition problem so that it correctly represented a given picture. However when shown a stick picture and subtraction problem, T made the changes as shown below.



At the beginning of the unit on subtraction T could not compute 5 - 1 without his fingers and even then made frequent errors. He might put up five fingers on one hand, one finger on another, take away the one and arrive at an answer of five. When 2-digit minus 2-digit problems were introduced starting with subtracting multiples of ten such as 70 - 30, T would show a set of 70 and a set of 30, take away the set of 30 and get an answer of 70.

T initially interpreted pictures such as: 

as 4 - 2 = 4. Since he had not memorized any subtraction facts and had a poor number sense, this did not seem unusual to him.

When the teacher showed a subtraction problem with objects, T could write the problem and the answer. This was surprising since the same task with addition caused him great difficulty. For a long time, if T did not have any structure, such as the place value chart, he did not know where to put things and had difficulty lining up the tens and ones. For example, he often made errors of the following kind:

$$\begin{array}{r} 38 \\ -\ 2 \\ \hline 18 \end{array}$$

After some practice T was able to interpret addition and subtraction pictures and write the corresponding problems, although the abacus pictures frequently confused him. When symbolic subtraction problems were first presented without pictures T was lost. He would add the subtrahend and minuend together, subtract the minuend, and get the subtrahend as an answer. He needed to use objects to work symbolic worksheets long after the pictures were eliminated, but often refused to do it because he was the only one who was using them.

Although T would sometimes forget to complete a problem, as in the following example:

$$\begin{array}{r} 45 \\ -\ 2 \\ \hline 3 \end{array}$$

he always read the numbers as whole numbers, not just digits and would always read the complete answer after finishing a problem. That is, T would read the problem as forty-five take away 2 equals . . . rather than as five take away 2 . . .

ADDITION AND SUBTRACTION WITH REGROUPING

Before receiving instructions on regrouping T was asked to solve an addition problem which required regrouping. His performance on this task is shown below:

$$\begin{array}{r} 27 \\ +35 \\ \hline 512 \end{array}$$

T read the completed problem as "27 + 35 equals fifty-twelve." He then said "50 + 12 leaves 3," evidently adding the one and two together. He then solved the problem with Dienes blocks and stated that "50 + 12 = 63." However, T changed his answer to 61. Such differences between T's performances at the enactive and the symbolic levels were not uncommon.

When given a subtraction problem requiring regrouping before instruction, T responded with the common error:

$$\begin{array}{r} 53 \\ -\ 24 \\ \hline 31 \end{array}$$

He tried to work the problem on the abacus but was not successful.

When shown these two addition problems:

$$\begin{array}{r} 24 \\ +\ 13 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 22 \\ +\ 19 \\ \hline \end{array}$$

and asked why the problem on the right was harder, T responded "Because it has a 9 in it." Similarly given two subtraction problems, T did not observe that the necessity for regrouping could make one of the problems more difficult.

Initially T needed to use the manipulative aids to work addition problems requiring regrouping. He would set up the problem with the manipulatives even if a picture was given, as the pictures tended to confuse him. As T became more proficient symbolically, when he was asked to write the problem represented by a picture he would first count everything in the picture, then count and write the addends, then work the problem strictly symbolically. The shading in the picture to show regrouping of 10 ones for a ten was meaningless to him. For example, T could solve a problem as shown:

$$\begin{array}{r} 24 \\ +\ 39 \\ \hline \end{array}$$

T had no difficulty remembering to put in the middle steps in the addition algorithm when regrouping was necessary. When pictures were present he would not use the middle steps to complete the answer, but would count the pictures. However, when it came to just written work he had no trouble using the sum of the ones and the tens to get the answer. T always added the ones first, usually saying out loud as he did it "First, add the ones . . . "

When shown the problem and solution:

$$\begin{array}{r} 34 \\ +\ 29 \\ \hline 513 \end{array}$$

and asked to read it, T said "it's 63." When asked if the answer was OK as written, he said "Yes. It's 63."

T took great pride in knowing the basic addition and subtraction facts and was the first in the group to commit many of them to memory. His errors in computation were frequently due to his guessing a fact which he though he "knew."

To prepare the children for regrouping in addition and subtraction practice was given trading 10 ones for one ten and vice versa with the manipulative aids. T was not convinced at the beginning that the number of the set stayed the same after trading. He would concentrate on the numerousness of the objects and thus would judge the set with the most ones to have more. After he finally accepted the equality he still had great difficulty verbalizing it, especially statements such as "2 tens and 4 ones equals 1 ten and 14 ones." It took T a long time to learn what kind of trading to do for "Make more ones." When they began to symbolize the trading, T initially insisted on putting the result after trading below the original tens and ones:

~~6~~ ~~4~~

5 14

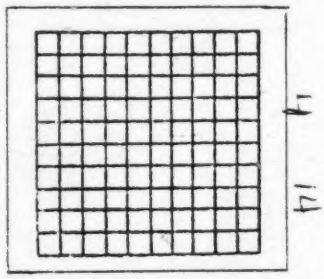
When recording the trading shown by pictures of objects before and after trading, T did not need to count the tens and ones in the after trading picture because he trusted that the total number had been preserved.

When given orally the first subtraction problem requiring regrouping, 45 - 17, T though it was impossible, even after displaying 45 with one of the manipulative aids. After some contemplation and probing by the teacher he said he though he needed to trade. His first inclination, however, was to add 5 ones to the 5 ones he already had so that he would have 10 ones. Although given any 2-digit number he was able to trade a ten for ten ones correctly with any of the manipulatives and show it symbolically; having the minuend present confused him. He found the three step process by which he (a) must decide whether to trade, (b) trade if necessary, and (c) perform the subtraction, very difficult at first. When pictures or objects were present his tendency was to just write the answer and not show any trading. He did not really master the symbolic algorithm until the manipulatives were abandoned. He frequently made this mistake when trading:

3	10
~~4~~	6

and also often traded when it was not necessary. He would, however, ignore the trading when actually working the problem if it had not been necessary in the first place, and would get the correct answer.

After a great deal of practice T became proficient with the subtraction algorithm and could demonstrate with almost all of the manipulatives with which he was familiar. Once when using the grid paper to solve 54 - 37, T shaded in the paper as shown below:



After shading in 54 and 37, he indicated the trading of a ten for ten ones by writing 4 and 14 next to the 54. When trying to subtract the 37 mentally he first subtracted 7 from 14 aloud, got 7, and then said the answer was 37.

When doing addition and subtraction problems mentally he would frequently compute the ones correctly, but forget to carry or borrow the tens. Thus he would compute

$$\begin{array}{r} 58 \\ +\ 25 \\ \hline \end{array}$$

and get 73, and

$$\begin{array}{r} 32 \\ -\ 8 \\ \hline \end{array}$$

and get 30. Occasionally he would revert back to the strategy he used before instruction on regrouping. For example, T worked this problem:

$$\begin{array}{r} 43 \\ -\ 26 \\ \hline 23 \end{array}$$

When asked to read it, he said "3 take away 6 leaves 3, 4 take away 2 leaves 2, so it's 23."

T initially did not recognize a problem like the one below as a subtraction impasse that necessitated trading:

$$\begin{array}{r} 30 \\ -\ 17 \\ \hline 27 \end{array}$$

He would again call on the "take the smaller number from the larger number" strategy.

(Without instruction) When asked to work 3-digit addition and subtraction problems which required regrouping, T exhibited little ability to transfer his knowledge of 2-digit algorithms to these new situations. When attempting to work

$$\begin{array}{r} 345 \\ +\ 173 \\ \hline \end{array}$$

he showed both addends correctly on the abacus and combined them. He did not trade, however, and could not give an answer. He said "That's 4 (pointing to the ten beads), that's 8 (pointing to the ones beads)." He eventually gave an answer of 51 which he computed mentally (and aloud) by adding 34 + 17.

His response to another 3-digit addition problem is given below:

$$\begin{array}{r} 426 \\ +\ 182 \\ \hline 508 \end{array}$$

After completing the problem he said "So it's 518." When asked how he got that he said "5 hundreds, 10 and 8 tens, so it's 518."

T could not even read 3-digit subtraction problems correctly, but instead treated them as three separate problems. He read

$$\begin{array}{r} 524 \\ -\ 183 \\ \hline \end{array}$$

as "5 take away 1 leaves 5, 2 take away 8 leaves 4, 4 take away 3 leaves 1, so it's 541." He tried to work it with the cubes. He put out 5 ten rods and took one away, but could not get an answer.

T performed better on 3-digit subtraction without the manipulatives. His response to such a purely symbolic situation is given below:

$$\begin{array}{r} 1\,13 \\ \not{2}\not{3}7 \\ -\ 176 \\ \hline 161 \end{array}$$

Although he computed the hundreds incorrectly, probably ignoring the trading, T's response to this situation reflects some transfer ability. It is interesting to note that he did not show trading symbolically on any 3-digit problem where use of the manipulatives was required. Recall that he did not master the 2-digit symbolic algorithm until he abandoned the manipulatives.

T's mathematics performance in the second grade far exceeded any expectations held by his classroom teacher. The increase in individual attention helped to keep him on task and forced him to complete work that normally he might have let slide. When working, he is diligent and thorough, although terribly slow. T has a firm background in place value and the addition and subtraction algorithms with numbers less than 100. He has indicated potential of understanding the algorithms as applied to larger numbers with a small amount of instruction. Unfortunately, returned to a class of 30 children, T's slow pace will be a great hindrance to his mathematical progress.

XV. THE CASE OF V

BY

CYNTHIA CLARKE

BIOGRAPHICAL AND TESTING INFORMATION

V is a female child born December 12, 1967. She lives with her mother, who is on welfare, four older brothers, aged nine to twenty-four years, a sister who is 23, and a number of other relatives. V was born prematurely, a fact which is often used by her mother to explain any learning difficulties V exhibits. The youngest of six children, V was babied at home although she developed a physical toughness about her in order to deal with four older brothers. She had problems with fighting early in her school career. V's personal hygiene was very poor and she was referred to the school nurse who visited her several times.

V was retained an extra year in the first grade. During that year, however, V was one of a few first graders placed in a second grade class due to space limitations. Thus the year of this study was actually V's second year in a second grade class. V was conscientious about doing her work. She indicated that she watched a great deal of television and also showed signs of religious influence.

V's test scores reflect below average competencies. Her Otis-Lennon IQ is 75. Her teacher considered V to be a low achiever in reading. The CTBS (Comprehensive Test of Basic Skills) battery, administered in the spring of the second grade, yielded a grade equivalent of 1.5 for reading, increased from .6 at the end of grade one. Her math grade equivalent showed the greatest increase, from .6 to 2.2, placing V in the 30th percentile. Her language and total scale grade equivalents were 1.8 and 1.7, respectively.

The KeyMath Diagnostic Arithmetic Test, administered at grade level 2.0, yielded a grade equivalent of 1.6. V scored below grade level on numeration

skills. She was unable to complete the simple pattern:

or to fill in the blanks ___, 6, 7, ___, 9; ___, 19; or 98, 99, ___, 101. She was able to perform simple addition (sums < 10) with pictures and symbolically. She could do only the simplest subtraction (subtrahend < 5) with pictures and symbolically. V could handle 1 + 1 and 2 + 2 mentally but could not mentally solve any problems involving more than two numbers. She was successful when performing horizontal addition (2 + 1 = ☐, 3 + 2 = ☐), but could not do the missing addend problems or the difficult numerical reasoning problems. She responded correctly to simple word problems with pictures.

The PMDC Grade Two Mathematics Test administered in the fall of the second grade showed V able to count picture sets of up to 13 numbers. She required prompting on all but one rote counting item, counting by tens. V exhibited no knowledge of 2-digit numerals and could not count forward from 35 or back from 44. She could only count to 30 by tens and could not count by tens starting at 26. She showed no knowledge of place value and could not determine or write the number of sets of straws or color-coded chips, or construct sets of a given number. She was able to use counters to solve simple addition and subtraction problems but could not handle sums greater than ten. V knew 12 was more than 8, but chose 19 as more than 31 and did not understand the meaning of "less." When asked to order cards with 5, 3, 9, and 2 on them from smallest to largest, she put them down like this:

5 3

2 6

The spring administration of the KeyMath Test (at grade level 2.8) yielded a grade equivalent of 2.2. V's numeration skills improved, with a score above third grade level. She now could perform two-digit addition with regrouping with sums less than 100. Her responses to both two-digit subtraction problems, with and without regrouping, were incorrect, but in each case she was one off the correct answer indicating a probable fact error resulting from an incorrect counting on strategy. V was able to do simple missing addend problems in horizontal form.

The spring administration of the PMDC Grade Two Mathematics Test indicated a great improvement in V's counting skills. She required prompting on only one rote counting item, counting by tens from 26, which she still could not do. She counted from 6 to 15, 35 to 46, back from 6 to 1 (skipping 4) and to 100 by tens. She could only count back to 40 from 44. She demonstrated understanding of place value by answering all the straw and color-coded chip items correctly, counting by tens and ones in each case. V used beans to correctly solve simple addition and subtraction problems (sums < 10), and counted on with beans to solve 18 + 5. She used beans to show 23 - 7, but miscounted the remaining set. She correctly answered all of the more, less, and ordering problems.

As a student in the T1 teaching experiment, V was assigned to the 2M group, using six manipulative aids: popsicle sticks, Unifix cubes, Dienes blocks, grid paper, abacus, and colored chips. According to her test scores, V was the lowest ability child of the six in the group. The remainder of this report will be divided into four sections: place value and numeration, ordering, addition and subtraction without regrouping, and addition and subtraction with regrouping.

PLACE VALUE AND NUMERATION

V could not perform any of the place value tasks on the PMDC Grade Two Mathematics Test at the beginning of the second grade. She knew nothing about numbers greater than 20 and could not read at all, thus she had difficulty when she encountered exercises such as:

2 longs + 3 units

____ + ____

_____

V had very little difficulty with two-digit numeration at the enactive and iconic levels, but proficiency at the symbolic level was very slow coming. She could write the multiples of ten when they were read orally, but frequently could not read a number which she had written. She seemed to understand the multiples of ten when relating to the manipulative aids, yet she completed a worksheet on this topic like this:

1 ten	10
2 tens	20
3 tens	30
4 tens	40
5 tens	50
6 tens	06
7 tens	07
8 tens	08
9 tens	09

When learning the numbers 20 through 30, V was slow to understand tens and ones. She would put out one long to show 26 and then count out units until she reached 26. She put out two red chips and ten white chips to show 30, responding that there are two tens in 30 when asked. After trading ten white chips for a red chip, she said there were three tens in 30, apparently seeing no contradiction with her previous statement.

It took V a long time to connect 30 and 3, 40 and 4, and so on, even at the verbal level. She would put out 6 bundles to show 42. After counting to 60 and realizing that was not correct, she would add more bundles, hoping eventually to reach "forty."

V frequently reversed digits when writing numerals in the range 13 - 19. V understood and could display two-digit numbers with manipulatives long before she could read or write them. In fact, if shown a numeral, she could display it correctly with a manipulative, then would count the objects to determine what the number was. During a taped interview she read both 35 and 53 as 25. The conversation which followed is given below:

Q: Why is this 25 (pointing to 53)?

V: Because a 5 and a 3.

Q: What does the 5 mean?

V: 5 tens.

Q: What does the 3 mean?

V: 3 ones.

It was obvious that V understood a great deal more than her ability to deal with the symbolization reflected. As part of the same interview V was shown the numeral 24 and told "I am thinking of 24 as 20 plus 4. Can you think of 24 in some other way?" V responded "2 plus 4." She completed an exercise dealing with the same concept as follows:

4 tens and 6 ones

4 and 6

46

V read 46 as 26.

While eventually V was able to display any number less than 100 with any of the six manipulative aids, she could not show 27 with cups of ten candies each and single pieces of candy.

V had considerable difficulty recalling what came after 190 when counting by tens. Questions such as "How many tens are in 170?" confused her. She made frequent errors on her worksheet. For example:

140

1 hundred, 14 tens

40 tens

or

140

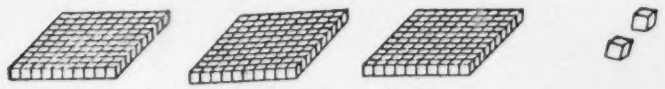
1 hundred, 40 tens

40 tens

V did not transfer her knowledge of writing two-digit numerals to writing numbers greater than 100. She had difficulty writing numbers in sequence especially where bridging of tens was involved. She repeatedly wrote 108, 109, 1010.

After counting by tens to 1000 as a group and discussing the hundreds, tens and ones in multiples of 100, V could not show 300 with bundles of sticks. She put out 13 bundles counted to 130, but was unable to correct herself. Even when she put 100 (10 bundles) in a pile in response to a hint, she still could not complete the task. She never understood the connection between 220 and 22, hence after displaying 22 tens with blocks she would have to count by tens to find out how many altogether. Similarly, if then asked how many tens in 220 she would recount the longs.

When evaluated on her knowledge of three-digit numbers, V could not write or circle the correct number of objects shown if there were only hundreds and ones present. An example of a typical response is shown below:



32

She had no trouble circling the objects represented by a particular digit. She could also construct a 3-digit numeral from a chart giving the number of hundreds, tens, and ones, in random order.

ORDERING

V entered the second grade with very little knowledge about more, less, or the ordering of the natural numbers. As her understanding of two-digit numeration increased she became quite proficient at ordering pictures. She quickly adopted the strategy of comparing the tens first when ordering sets of pictures such as those representing 7, 23, 34, 43, 75, 90. She made occasional errors, such as putting the 7 just before 75 obviously interpreting it as 70. She also would sometimes concentrate on numerousness as when she ordered this set: 20, 21, 13, 14, 50, 51, 52, 99. However, when it appeared to her that only one decade was involved she would count two or three of the pictures completely then begin looking only at the ones for the purpose of ordering them. In contrast to the other children, V performed better when the manipulative aids shown in the pictures were varied forcing her to count instead of making impulsive judgments from the patterns alone.

V acquired the concepts more and less rapidly although the symbols > and < never became meaningful to her. Given two numbers less than 100 V could make correct verbal "more than" and "less than" statements about them, but constructing a sentence using one of the symbols resulted in what appeared to be guesswork. Once when comparing the numbers 69 and 96, she put the cards with the numerals on them on the board in front of her and said "These are the same, see? Nine and nine, six and six (pointing at the respective digits). When she was asked to read the numbers, however, she realized that 96 was larger. When comparing 56 and 52, V said 56 is more "because 6 is more than 2." She also said 27 was less than 41 but gave as her reason "because 7 is less than."

V was slow to understand one more (less) and ten more (less) but eventually could show them with any of the six manipulatives and verbally give the result of adding (taking away) one or ten more (less) to a given number. She could rarely give the verbal response without objects present, especially one or ten less. When confronted with an unfamiliar manipulative (cups of candy) V was unable to show one more or ten more. However when one more and ten more were added, she was able to give the resulting new numbers.

When dealing with numbers greater than 100, V again needed objects present to start at a given number and count by ones, tens, or hundreds, especially when bridging of tens or hundreds was required. At the end of the year V was shown a three-digit numeral and told to "Write the number which is one more than this number." Similarly she was asked to write one less, ten more(less), and one hundred more (less) than given numbers. V obviously did not understand what she was being asked to do and simply copied the hundreds or tens digit in the blank. Later she was instructed to put out objects corresponding to the numeral in the first item and then to show ten more and write how many. V did this correctly and was then able to respond correctly to the other five items without using manipulatives. It was necessary that she visualize the concrete interpretation of the task before she could perform it in the abstract.

ADDITION AND SUBTRACTION WITHOUT REGROUPING

V was slow to observe the "2 + 5 = 7, so 20 + 50 = 70" pattern at the oral level, but observed that

$$\begin{array}{r} 30 \\ +\ 40 \\ \hline 70 \end{array}$$

was just like

$$\begin{array}{r} 3 \\ +\ 4 \\ \hline 7 \end{array}$$

and questioned why the zeros had to be there.

When matching pictures with addition problems, V would add the ones in the problem, find a picture with the correct number of ones altogether, and then worry about the tens.

V did not connect the verbal drill on the three equal expressions:

43 = 4 tens + 3 ones = 40 + 3, with the written exercise even though she was able to make the oral statements.

V had difficulty determining the addends of a problem represented by a pictured display. When she could figure them out, she still had trouble deciding where to put them unless the structure + ____was given to indicate when to write. After completing a problem, V could rarely say what problem it was that she had just worked or answered, for example, "So, what is 23 + 14?" after having worked it symbolically.

After being introduced to the short form, V frequently refused to put in the middle step when regrouping was not necessary. She preferred:

$$\begin{array}{r} 23 \\ +\ 14 \\ \hline 37 \end{array} \qquad \text{to} \qquad \begin{array}{r} +\ 23 \\ \quad 14 \\ \hline 7 \\ 30 \\ \hline 37 \end{array}$$

She would sometimes put the middle step in, but would not use it to find the answer.

V needed pictures or manipulatives in order to perform at the symbolic level for a long time, much longer than the other children. She was very inconsistent with symbolic addition. After working three or four problems correctly she might add one of the tens digits in with the ones, or add all the digits together. Once when adding:

$$\begin{array}{r} 1 \\ +\ 38 \\ \hline \end{array}$$

she answered 11, adding the 3 and 8 together. She also insisted on one occasion that the two fives in

$$\begin{array}{r} 4 \\ +55 \\ \hline \end{array}$$

were both ones. The one-digit addend on top consistently confused her.

Problems with zeroes caused her difficulty. She could answer problems such as 60 + 20 orally but the zeroes seemed to confuse her when working the same problem symbolically.

In February after the unit on two-digit addition and subtraction without regrouping, V gave the following responses:

$$\begin{array}{r} 5 \\ +23 \\ \hline 208 \end{array}$$

Q: Read the problem for me.

V: 20 + 5 + 3 is 23.

Q: How did you get the answer?

V: I had 20 and 5 and 3.

$$\begin{array}{r} 30 \\ +40 \\ \hline 50 \end{array}$$

Q: How did you get your answer?

V: 3 and 4 more is 50, and zero ones.

$$\begin{array}{r} 42 \\ +\ 7 \\ \hline 409 \end{array}$$

Q: Read the problem for me.

V: 40 and 2 and 7 is 90.

These responses were inconsistent with her classwork which was usually correct except for an occasional fact error.

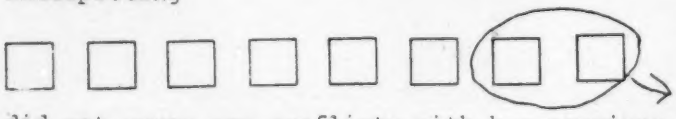
V could not do 24 + 32 mentally, but counted on to correctly solve 23 + 5 mentally. V was successful when solving addition problems with any of the six manipulatives, but when using beans, some in bags of 10, V used single beans to represent both tens and ones. She did, however, arrive at the correct answer.

When asked to compare a picture showing 25 + 2 with the problem

$$\begin{array}{r} 25 \\ +\ \ 2 \\ \hline 45 \end{array}$$

V said the problem goes with the picture. She merely compared the addends but did not take the whole problem into consideration. Such oversights were common when V was asked to reflect on something already completed.

When presented with the very first subtraction problem verbally, 7 - 3, V made a set of 7 and a set of 3, took away the set of 3 and got an answer of 7. Her number sense was very poor at the beginning, so that interpreting



as 6 - 2 = 6

did not cause any conflicts with her previous knowledge.

When working symbolic subtraction problems involving only multiples of ten, such as 40 - 10, V would show 4 - 1 on her fingers, saying as she did it "40 take away 10 is 30."

As with addition, V was lost without some structure, at least - ____, and had trouble lining things up. This difficulty caused errors such as the following:

$$\begin{array}{r} 38 \\ -\ 2 \\ \hline 18 \end{array}$$

When a worksheet required her to look at a symbolic problem and draw rings on the pictured display around the objects to be taken away, V would do the ones first, write the difference, then do the same for the tens.

When asked to solve:

$$\begin{array}{r} 45 \\ -\ 12 \\ \hline \end{array}$$

with cups of candy and single pieces of candy, V computed the answer first then put out three cups of 10 candies each, then three more cups of ten each and said "33." She could, however, perform two-digit subtraction without regrouping with any of the six familiar manipulatives.

ADDITION AND SUBTRACTION WITH REGROUPING

When presented with an addition problem requiring regrouping before receiving instruction on the regrouping algorithm, V made the common error:

$$\begin{array}{r} 27 \\ +\ 35 \\ \hline 512 \end{array}$$

She could not read the whole problem but read the answer as "fifty-twelve." When asked how she got her answer, she responded "I added 20 and 30 and got 50, and I added 7 and 5 and got 12." When she worked the problem with Dienes blocks, she counted the five longs, "10, 20, 30, 40, 50," and then the 12 units "51, 52, . . ., 58, 59, 60, 70, 80."

While V was able to trade to make fewer ones with any of the six manipulatives, she did not use this technique when solving regrouping problems, but simply counted all the ones until she reached an answer. She showed

$$\begin{array}{r} 58 \\ +\ 25 \\ \hline \end{array}$$

with sticks but could not give an answer. Instead of counting all the tens first, she counted on from 25 incorrectly as follows: "25, 35, 30, 35, 55, 75, 85, 95, 96, 97, 98, 99" and stopped.

V needed objects or pictures for a long time before she felt comfortable performing addition with regrouping symbolically. After she mastered the algorithm she only used the pictures to determine the addends. The shaded loop of ten showing regrouping was meaningless to her.

V would put in the middle step on addition problems only when necessary, frequently working two consecutive problems on a worksheet differently if one required regrouping and one did not, as shown below:

```
  34          31
 +28         +25
  12          56
 +50
  62
```

After dealing with two-digit plus two-digit addition, V found problems with one-digit addends confusing. She worked

```
  46
+  7
```

with beans by using single beans for each digit instead of using bags of 10 beans for the tens. Thus she arrived at an answer of 17.

While proficient with the addition algorithm, when given a single problem during an evaluation interview she would occasionally forget. A description of one such instance follows:

```
  34
+ 29
 513
```

Q: Read the problem for me.

V: It's 54.

Q: Why should the answer be 54?

V: Because you add the one and three together.

When given a subtraction problem requiring regrouping before instruction on the algorithm, V responded as follows:

```
  53
- 24
  30.
```

She showed this with cubes and just ignored the subtraction impasse.

V was not convinced at first that trading ten ones for one ten or vice versa did not change the total number. Even after she began believing this she still had difficulty verbalizing the inequality, such as 2 tens and 3 ones equals 1 tens and 13 ones. When told to "Trade to make ones" she was often confused, not knowing whether to trade one ten for ten ones or vice versa.

When she began symbolizing the trading she was doing with objects, V did not want to cross out to show the trading because it looked "messy." She actually made very few subtraction errors due to looking at the original digits and eventually conceded to the crossing out technique.

On the worksheet of pictures showing displays before and after trading, V recounted everything after trading, still not trusting that the total number had been preserved. Although V could eventually trade one ten for ten ones correctly, and show this symbolically, she found it difficult to use this skill when it actually was necessary in subtraction problems. When pictures were present she usually refused to show trading.

When writing a problem for pictures, V completely ignored the middle trading picture and used the other two to set up the problem and get the answer. Eventually she used the algorithm to obtain the answer, ignoring both the second and third pictures.

When worked addition and subtraction problems on the same worksheet, V sometimes got confused and developed a third "combination algorithm."

$$\begin{array}{r} 75 \\ -\ 38 \\ \hline 13 \\ 40 \\ \hline 53 \end{array}$$

V was unable to show how to solve the above problem with Dienes blocks.

After mastering the subtraction algorithm, V was still rather inconsistent. She would occasionally revert to the "take the smallest from the largest" technique, as she did in the problem below:

$$\begin{array}{r} 43 \\ -\ 26 \\ \hline 23 \end{array}$$

When asked to read the problem she said "3 take away 6 leaves 3; 4 take away 2 leaves 2."

When presented with 3-digit addition and subtraction problems without instruction, V interpreted each one as three separate problems. For example, she read:

$$\begin{array}{r} 345 \\ +\ 173 \\ \hline \end{array}$$

as "3 and 1 is 4; 4 and 7 is." The interviewer stopped her saying "Just read the problem." She continued "3 and 1, 4 and 7, 5 and 3." This behavior was consistent throughout all the problems, addition and subtraction alike.

V was unable to transfer her knowledge of the two-digit regrouping algorithms to three-digit situations. She added

$$\begin{array}{r} 156 \\ +\ 178 \\ \hline \end{array}$$

with chips, ignored the hundreds, and counted to 133 without trading.

When working

$$\begin{array}{r} 245 \\ -\ 148 \\ \hline \end{array}$$

she said "2 take away 1 is 1" (putting out one white chip) then "5 take away 8 is 3." When asked what the answer is, V said "3."

V seemed to have a fair understanding of tens and ones and the two-digit algorithms. She needed more work at the enactive level to better prepare her for the symbolic. Perhaps this would have aided in her transfer to three-digit situations.